

The world according to Rényi: Thermodynamics of multifractal systems

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Abstract

We discuss basic statistical properties of systems with multifractal structure. This is possible by extending the notion of the usual Gibbs–Shannon entropy into more general framework - Rényi’s information entropy. We address the renormalization issue for Rényi’s entropy on (multi)fractal sets and consequently show how Rényi’s parameter is connected with multifractal singularity spectrum. The maximal entropy approach then provides a passage between Rényi’s information entropy and thermodynamics of multifractals. Important issues such as Rényi’s entropy versus Tsallis–Havrda–Charvat entropy and PDF reconstruction theorem are also studied. Finally, some further speculations on a possible relevance of our approach to cosmology are discussed.

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I. INTRODUCTION

The past two decades have witnessed an explosion of activity and progress in both equilibrium and non-equilibrium statistical physics. The catalyst has been the massive infusion of ideas from information theory, theory of chaotic dynamical systems, theory of critical phenomena, and quantum field theory. These ideas include the generalized information measures, quasi-periodic and strange attractors, fully developed turbulence, percolation, renormalization of large-scale dynamics, and attractive, albeit speculative, ideas about quark–gluon plasma formation and dynamics. It is the purpose of this paper to proceed in this line of development. The issue at the stake is what modifications in statistical physics one should expect when dealing with systems with varied fractal dimension - multifractals. The view which we present here hinges on two mutually interrelated concepts, namely on Rényi’s information entropy [3,4] and (multi)fractal geometry. In this connection we would like to stress that in order to exhibit the link between Rényi information entropies and (multi)fractal systems as generally as possible we do not put much emphasize on the important yet rather narrow class of (multi)fractal systems - chaotic dynamical systems.

One of the fundamental observations of information theory is that the most general functional form for the mean transmitted information (i.e., information entropy) is that of Rényi. In Section II we briefly outline Rényi’s proof and discuss some fundamentals from information

theory which will show up to be indispensable in following sections. We show that with certain mathematical cautiousness Shannon’s entropy can be viewed as a special example of Rényi’s entropy in case when Rényi’s parameter $\alpha \rightarrow 1$. We also address the question of the status of Tsallis–Havrda–Charvat (THC) entropy [1,2] in the framework of information theory.

Although Rényi’s information measure offers very natural - and maybe conceptually the cleanest - setting for the entropy, it has not found so far as much applicability as Shannon’s (or Gibbs’s) entropy. The explanation, no doubt, lies in two facts; ambiguous renormalization of Rényi’s entropy for non-discrete distributions and little insight into the meaning of Rényi’s α parameter. Surprisingly little work has been done towards understanding both of the former points. In Section III we aim to address the first one. We choose, in a sense, a minimal renormalization prescription conforming to the condition of additivity of independent information. Rényi’s entropy thus obtained is then directly related to the information content (“negentropy”).

To clarify the position of Rényi’s entropy in physics, or in other word, to find the physical interpretation for α parameter, we resort in Section IV to systems with a multifractal structure. Such systems are very important and highly diverse, including the turbulent flow of fluids [5,6], percolations [7], diffusion-limited aggregation (DLA) systems [8], DNA sequences [9], finance [10], and string theory [11]. Using the reconstruction theorem we

argue that in order to obtain a “full” information about a (multi)fractal system we need to know Rényi’s entropies to all orders. Still, for discrete spaces and simple metric spaces (like \mathbb{R}^d) we find that the contribution from Shannon’s entropy dominates over all other Rényi entropies. We further show that from the maximal entropy (Max-Ent) point of view, extremizing the Shannon entropy on a multifractal is equivalent to extremizing directly Rényi’s entropy without invoking the multifractal structure explicitly. Application of this result to a cosmic strings network will be presented elsewhere [12].

We close with Section V where we present some speculations on the relevance of the outlined approach to string cosmology and quantum mechanics. For reader’s convenience we supplement the paper with eight appendices which clarify some finer mathematical manipulations.

II. RENYI’S ENTROPY OF DISCRETE PROBABILITY DISTRIBUTIONS

A. Rényi’s entropy and information theory

We begin this section by summarizing the information theory procedure leading to Rényi’s entropy [3,4]. This is of course well known but it may be useful to repeat it here in order to make our discussion self-contained. We will also need to generalize it when considering THC entropy in Section IID and axiomatization of Rényi’s entropy in Appendix B.

Let us start with a discrete probability distribution $\mathcal{P} = \{p_1, p_2, \dots, p_n\}$ fulfilling usual conditions

$$p_k \geq 0, \quad \sum_k p_k = 1. \quad (2.1)$$

We then assume three things about information. Firstly, information should be additive for two independent events. Secondly, information should purely depend on \mathcal{P} . These two conditions can be also formulated in the following way: if we observe the outcome of two independent events with respective probabilities p and q , then the total received information is the sum of two partial ones. Therefore the following functional equality holds:

$$\mathcal{I}(pq) = \mathcal{I}(p) + \mathcal{I}(q). \quad (2.2)$$

The latter is well known modified Cauchy’s functional equation [13] which has (under fairly broad assumptions [4,14]) unique class of solutions - $\kappa \log_2(\dots)$. The constant κ is then fixed via appropriate “boundary” condition. Setting $\mathcal{I}(1/2) = 1$ we obtain the, so called, *Hartley measure* of information [15]. So the amount of information received by learning that event of probability p took place equals

$$\mathcal{I}(p) = -\log_2(p). \quad (2.3)$$

The third assumption is that if different amounts of information occur with different probabilities, the total amount of information is the average of the individual information weighted by the probabilities of their occurrences. In general, if the possible outcomes of an experiment are $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ with corresponding probabilities p_1, p_2, \dots, p_n , and \mathcal{A}_k conveys \mathcal{I}_k bits of information, then the total amount of information conveyed would be

$$\mathcal{I}(\mathcal{P}, \mathfrak{I}) = \sum_{k=1}^n p_k \mathcal{I}_k, \quad (2.4)$$

where $\mathfrak{I} = \{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_n\}$. However, the linear averaging implemented in (2.4) is only a specific case of a more general mean. If f is a real function having an inverse f^{-1} then the number

$$f^{-1} \left(\sum_k^n p_k f(x_k) \right), \quad (2.5)$$

is called the mean value of x_1, x_2, \dots, x_n associated with f . As shown in Refs. [16–18], (2.5) prescribes the most general mean compatible with postulates of probability theory (see, eg., [3]). The function f is often referred to as Kolmogorov–Nagumo’s function.

Former analysis suggests that in the most general case the measure of the amount of transmitted information should admit the form

$$\mathcal{I}(\mathcal{P}, \mathfrak{I}) = f^{-1} \left(\sum_{k=1}^n p_k f(-\log_2(p_k)) \right). \quad (2.6)$$

The natural question arises, what is the possible mathematical form of f , or in other words, what is the most general class of functions f which will still provide a measure of information compatible with the additivity postulate. Obviously for a given set of outcomes, many possible means can be defined, depending on which features of the outcomes are of interest. It comes therefore as a pleasant surprise to find that the additivity postulate allows only for two classes of f ’s - linear and exponential functions. The proof of this statement is simple and clarifies a good deal about f so for the future reference we sketch its main points. Alternative proof based on scaling argumentation is presented in Appendix A.

Let an experiment \mathcal{K} be a union of two independent experiments \mathcal{K}_1 and \mathcal{K}_2 . Let further assume that we receive $\mathcal{I}_k^{(1)}$ bits of information with probability p_k connected with \mathcal{K}_1 and $\mathcal{I}_l^{(2)}$ bits of information with probability q_l connected with \mathcal{K}_2 . As a result we receive $\mathcal{I}_k^{(1)} + \mathcal{I}_l^{(2)}$ bits of information with probability $p_k q_l$. We assume further that there is m possible outcomes in \mathcal{K}_1 experiment (i.e., $k = 1, 2, \dots, m$) and n possible outcomes in \mathcal{K}_2 experiment (i.e., $l = 1, 2, \dots, n$). Invoking the postulate of additivity we may write

$$\begin{aligned}
& f^{-1} \left(\sum_k^m \sum_l^n p_k q_l f \left(\mathcal{I}_k^{(1)} + \mathcal{I}_l^{(2)} \right) \right) \\
&= f^{-1} \left(\sum_k^m p_k f \left(\mathcal{I}_k^{(1)} \right) \right) + f^{-1} \left(\sum_l^n q_l f \left(\mathcal{I}_l^{(2)} \right) \right). \quad (2.7)
\end{aligned}$$

The former must hold completely generally irrespective of our choice of $\mathcal{P} = \{p_1, \dots, p_m\}$, $\mathcal{Q} = \{q_1, \dots, q_n\}$ and irrespective of the actual choice of independent experiments $\mathcal{K}_1, \mathcal{K}_2$. So if we choose $\mathcal{I}_l^{(2)} = \mathcal{I}$ independently of k we obtain from (2.7)

$$\begin{aligned}
& f^{-1} \left(\sum_k^m p_k f \left(\mathcal{I}_k^{(1)} + \mathcal{I} \right) \right) \\
&= f^{-1} \left(\sum_k^m p_k f \left(\mathcal{I}_k^{(1)} \right) \right) + \mathcal{I}. \quad (2.8)
\end{aligned}$$

Let us denote $f_y(x) = f(x + y)$ (so namely $f^{-1}(x) - y = f_y^{-1}(x)$). Thus Eq.(2.8) may be recast into the form

$$f_{\mathcal{I}}^{-1} \left(\sum_k^m p_k f_{\mathcal{I}} \left(\mathcal{I}_k^{(1)} \right) \right) = f^{-1} \left(\sum_k^m p_k f \left(\mathcal{I}_k^{(1)} \right) \right). \quad (2.9)$$

So functions $f_{\mathcal{I}}$ and f generate the same mean. It is well known in theory of means (see eg., [19]) that Eq.(2.9) holds only if $f_{\mathcal{I}}$ is a linear function of f . So we have

$$f_{\mathcal{I}}(z) = f(z + \mathcal{I}) = a(\mathcal{I}) f(z) + b(\mathcal{I}). \quad (2.10)$$

Here $a(\dots) \neq 0$ and $b(\dots)$ are independent of z . Without loss of generality we shall assume that $f(0) = 0$ (or otherwise we adjust b). As a result $b(\mathcal{I}) = f(\mathcal{I})$. Therefore

$$\begin{aligned}
f(z + \mathcal{I}) &= a(\mathcal{I}) f(z) + f(\mathcal{I}) \\
f(z + \mathcal{I}) &= a(z) f(\mathcal{I}) + f(z), \quad (2.11)
\end{aligned}$$

where the second line was obtained by a simple interchange $z \leftrightarrow \mathcal{I}$. Subtraction of both lines in (2.11) leads to the following separation of variables ($z \neq 0, \mathcal{I} \neq 0$):

$$\frac{a(z) - 1}{f(z)} = \frac{a(\mathcal{I}) - 1}{f(\mathcal{I})} = \gamma. \quad (2.12)$$

(γ is a constant independent both of z and \mathcal{I}). The solution of (2.12) has a simple form

$$a(x) - 1 = \gamma f(x). \quad (2.13)$$

Note that (2.13) holds true also for $x = 0$. In connection with (2.13) it is useful to distinguish two cases; $\gamma = 0$ and $\gamma \neq 0$. In the first case when $\gamma = 0$, $a(x) = 1$ and we get Cauchy's functional equation [13]

$$f(z + \mathcal{I}) = f(z) + f(\mathcal{I}), \quad (2.14)$$

which for $z, \mathcal{I} \in \mathbb{R}$ has the well known solution: $f(x) = cx$ with the nonzero constant c . This is in a sense the

most elementary Kolmogorov–Nagumo function. Plugging the latter into Eq.(2.6) the measure of transmitted information boils down to Shannon's measure

$$\mathcal{I}(\mathcal{P}, \mathfrak{S}) = - \sum_{k=1}^n p_k \log_2(p_k) \equiv \mathcal{H}. \quad (2.15)$$

In the second case when $\gamma \neq 0$, $a(x)$ fulfills the modified Cauchy's functional equation [13]

$$a(z + \mathcal{I}) = a(z)a(\mathcal{I}), \quad (2.16)$$

which for continuous $a(\dots)$ and $z, \mathcal{I} \in \mathbb{R}$ has only exponential solutions. Thus we may generally write: $a(x) = 2^{(1-\alpha)x}$ with $\alpha \neq 1$ being some constants. As a result we get $f(x) = [2^{(1-\alpha)x} - 1]/\gamma$. Plugging this into Eq.(2.6) the measure of transmitted information will be

$$\mathcal{I}_{\alpha}(\mathcal{P}, \mathfrak{S}) = \frac{1}{(1-\alpha)} \log_2 \left(\sum_{k=1}^n p_k^{\alpha} \right). \quad (2.17)$$

The information measure (2.17) is usually called the *generalized information measure* or *information measure of order α* , or simply *Rényi's entropy*. We will denote the explicit order of Rényi's entropy as a subscript in $\mathcal{I}(\dots)$.

Although the foregoing *operational* (pragmatic) way of arguing is quite robust, some readers may find more justifiable to see Rényi's entropy properly axiomatized. Actually, the Shannon entropy was firstly axiomatized by Shannon [20] and then later some axioms were weakened (or substituted) by Fadeev [21], Khinchin [22] and several other authors [23]. The Rényi entropy was axiomatized by Rényi himself [3,4] and afterwards sharpened by Daróczy [24] and others [25]. In further considerations we will find, however, useful to use a slightly different set of axioms than those utilized in [3,4,24,25]. In fact, in Appendix B we show that the information measures (2.15) and (2.17) can be characterized by the following axioms:

1. For a given integer n and given $\mathcal{P} = \{p_1, p_2, \dots, p_n\}$ ($p_k \geq 0, \sum_k^n p_k = 1$), $\mathcal{I}(\mathcal{P})$ is a continuous with respect to all its arguments.
2. For a given integer n , $\mathcal{I}(p_1, p_2, \dots, p_n)$ takes its largest value for $p_k = 1/n$ ($k = 1, 2, \dots, n$) with the normalization $\mathcal{I}(\frac{1}{2}, \frac{1}{2}) = 1$.
3. For a given $\alpha \in \mathbb{R}$; $\mathcal{I}(\mathcal{A} \cap \mathcal{B}) = \mathcal{I}(\mathcal{A}) + \mathcal{I}(\mathcal{B}|\mathcal{A})$ with

$$\mathcal{I}(\mathcal{B}|\mathcal{A}) = f^{-1} \left(\sum_k \varrho_k(\alpha) f(\mathcal{I}(\mathcal{B}|\mathcal{A} = \mathcal{A}_k)) \right),$$

and $\varrho_k(\alpha) = (p_k)^{\alpha} / \sum_k (p_k)^{\alpha}$ (distribution \mathcal{P} corresponds to the experiment \mathcal{A}).

4. f is invertible and positive in $[0, \infty)$.
5. $\mathcal{I}(p_1, p_2, \dots, p_n, 0) = \mathcal{I}(p_1, p_2, \dots, p_n)$, i.e., adding an event of probability zero (impossible event) we do not gain any new information.

B. Some observations about Rényi's entropy

Before going further let us observe some key characteristics of Rényi's entropy which will prove essential in following sections.

(a) $\mathcal{I}_\alpha(\mathcal{B}|\mathcal{A})$ appearing in the axiom 3 can be viewed as conditional information. In fact, in Appendix C we show that $\mathcal{I}_\alpha(\mathcal{B}|\mathcal{A}) = 0$ iff outcome \mathcal{A} uniquely determines outcome \mathcal{B} . We also show that when \mathcal{A} and \mathcal{B} are independent then $\mathcal{I}_\alpha(\mathcal{B}|\mathcal{A}) = \mathcal{I}_\alpha(\mathcal{B})$ and hence $\mathcal{I}_\alpha(\mathcal{A} \cap \mathcal{B}) = \mathcal{I}_\alpha(\mathcal{A}) + \mathcal{I}_\alpha(\mathcal{B})$, as expected. Alas the reverse implication (i.e., $\mathcal{I}_\alpha(\mathcal{B}|\mathcal{A}) = \mathcal{I}_\alpha(\mathcal{B}) \Rightarrow \mathcal{A}$ and \mathcal{B} are independent) generally holds only when \mathcal{B} has uniform distribution.

(b) It is interesting to note that we can write (with a bit of hindsight) in the axiom 3

$$\mathcal{I}_\alpha(\mathcal{B}|\mathcal{A}) = f^{-1} \left(\sum_k \varrho_k(\alpha) f(\mathcal{I}_\alpha(\mathcal{B}|\mathcal{A} = \mathcal{A}_k)) \right).$$

Similarly, we can write Eq.(2.6) as

$$\mathcal{I}(\mathcal{P}) = f^{-1} \left(\sum_k \varrho_k(1) f(\mathcal{I}_1(\mathcal{A} = \mathcal{A}_k)) \right).$$

This indicates that when the constituent information of order α enter a mean value calculation they must be weighted by $\varrho_k(\alpha)$'s and not p_k 's, and this should hold true whatever the Kolmogorov–Nagumo function is. The former result may be generalized in the following way: Whenever outcomes of a measurement carry an information of order α they must be weighted with $\varrho_k(\alpha)$. When outcomes actually carry information of order α will be discussed in Section IV B.

(c) Another important property of Rényi's entropy is its concavity for $\alpha < 1$ (for $\alpha > 1$ Rényi's entropy is not purely convex nor purely concave). This a simple consequence of the fact that both $\log_2(x)$ and x^α ($\alpha < 1$) are concave functions (while x^α is convex for $\alpha > 1$).

(d) A notable point which we will use in Section IV is that \mathcal{I}_α is a monotonous decreasing function of α . This might be seen from the inequality

$$\begin{aligned} \frac{d\mathcal{I}_\alpha}{d\alpha} &= \frac{1}{(1-\alpha)^2} \{ -\log_2 \langle \mathcal{P}^{1-\alpha} \rangle_\alpha + \langle \log_2 \mathcal{P}^{1-\alpha} \rangle_\alpha \} \\ &\leq 0. \end{aligned} \quad (2.18)$$

Here the expectation value $\langle \dots \rangle_\alpha$ is defined with respect to the distribution $\varrho_k(\alpha)$. The last line of (2.18) is due to Jensen's inequality and due to concavity of $\log_2(x)$. Note that $d\mathcal{I}_\alpha/d\alpha = 0$ only when the Jensen inequality used in the derivation (2.18) is an equality. This happen iff $\mathcal{P} = \text{const.}$ (see e.g., [19]), or in other words when \mathcal{P} is uniform. Consequently either \mathcal{I}_α is a strictly monotonous decreasing function of α or all \mathcal{I}_α are identical. One never finds, for example, $\mathcal{I}_{\alpha_1} < \mathcal{I}_{\alpha_2} = \mathcal{I}_{\alpha_3}$ for $\alpha_1 > \alpha_2 > \alpha_3$.

C. Rényi's entropy and Shannon's entropy

Now we turn to the investigation of the information measure of order 1. An important element in this discussion is the fact that \mathcal{I}_α is analytic in $\alpha = 1$. This can be seen by continuing the index α into the complex plane and inspecting the behavior of $\log_2(\sum_{k=1}^n p_k^z)$ for $z \in \mathbb{C}$. The former is analytic provided that $\sum_{k=1}^n p_k^z$ is not laying on the negative real axis. Let us now consider the situation where $z = 1 + r e^{i\varphi}$ (i.e., we draw a circle with the radius r centered at $z = 1$). Thus $\log_2(\sum_{k=1}^n p_k^z)$ is analytic throughout the entire complex plane except the regions where the following two conditions hold

$$\begin{aligned} \sum_{k=1}^n \sin(r \sin \varphi \ln(p_k)) &= 0, \\ \sum_{k=1}^n p_k^{r \cos \varphi + 1} \cos(r \sin \varphi \ln(p_k)) &\leq 0. \end{aligned} \quad (2.19)$$

Let us put $r < |\pi/(2 \ln(p_k)_{\min})|$. Then evidently for such r 's the conditions (2.19) cannot be fulfilled together and we are safely in the analyticity region. Consider the contour integral

$$I = \oint dz \frac{\log_2(\sum_{k=1}^n p_k^z)}{1-z} = \oint dz \mathcal{I}_z(\mathcal{P}), \quad (2.20)$$

around a contour $z = 1 + r e^{i\varphi}$, $\varphi \in [0, 2\pi)$. The residue theorem assures then that (2.20) vanishes and as a result Rényi's entropy is analytic everywhere inside the contour (so also at $z = 1$). This shows that the singularity of $\mathcal{I}_\alpha(\mathcal{P})$ at $\alpha = 1$ is only spurious and, in fact, Rényi's entropy is differentiable at $\alpha = 1$ to all orders. Using the Cauchy formula we can directly write

$$\begin{aligned} \mathcal{I}_1(\mathcal{P}) &= \frac{i}{2\pi} \oint dz \frac{\log_2(\sum_{k=1}^n p_k^z)}{(1-z)(1-z)} \\ &= \frac{1}{2\pi i} \oint dz \left(\frac{d}{dz} \frac{1}{(z-1)} \right) \log_2 \left(\sum_{k=1}^n p_k^z \right) \\ &= \frac{i}{2\pi} \oint dz \frac{\sum_{k=1}^n p_k^z \log_2(p_k)}{(z-1) \sum_{k=1}^n p_k^z} \\ &= - \sum_{k=1}^n p_k \log_2(p_k) = \mathcal{H}(\mathcal{P}), \end{aligned} \quad (2.21)$$

where the contour of integration is the same as in the case (2.20). It is usually argued that it is a matter of modification of one of Shannon's axioms to get Rényi's entropy. We, however, do not intend to follow this path simply because the Shannon entropy, as we have just seen, can be uniquely determined from the behavior of (analytically continued) Rényi's entropy in the vicinity of $z = 1$. In fact, we even do not need to be in the vicinity because the circle used in the contour integral (2.21) can be analytically continued to any curve which lies in the 1st and 4th quadrant and which encircles the point $z = 1$. View

which we intend to advocate here is that the Shannon entropy is not a special information measure deserving separate axiomatization but a member of a wide class of entropies embraced by a single unifying axiomatics.

An important consequence of the fact that \mathcal{I}_α is a monotonous decreasing function of α is embodied in the following two inequalities

$$\mathcal{H} < \mathcal{I}_\alpha < \log_2 n, \quad 0 < \alpha < 1, \quad (2.22)$$

$$\mathcal{I}_\alpha < \mathcal{H} < \log_2 n, \quad \alpha > 1. \quad (2.23)$$

Inequality (2.23) shows that \mathcal{H} represents an upper bound for all Rényi entropies with $\alpha > 1$. This finding will play an important rôle in the reconstruction theorem in Section IV B.

D. Rényi's entropy and THC entropy

Due to an increasing interest in long-range correlated systems and non-equilibrium phenomena there has been currently much discussed the, so called, Tsallis (or non-extensive) entropy. Although firstly introduced by Havrda and Charvat in the cybernetics theory context [1] it was Tsallis [2] who exploited its non-extensive features and placed it in a physical setting. THC entropy reads

$$\mathcal{S}_\alpha = \frac{1}{(1-\alpha)} \left[\sum_{k=1}^n (p_k)^\alpha - 1 \right], \quad \alpha > 0. \quad (2.24)$$

The most important properties of THC entropy can be easily read out of (2.24). For instance, employing Jensen's inequality we have for $\alpha > 1$ that $\sum_k p_k^\alpha \leq 1$ (while for $0 < \alpha < 1$ the reverse inequality holds) and hence \mathcal{S}_α is non-negative. Similarly, choosing any pair of distributions \mathcal{P} and \mathcal{Q} , and a real number $0 \leq \lambda \leq 1$ we have

$$\mathcal{S}_\alpha(\lambda\mathcal{P} + (1-\lambda)\mathcal{Q}) = \lambda\mathcal{S}_\alpha(\mathcal{P}) + (1-\lambda)\mathcal{S}_\alpha(\mathcal{Q}), \quad (2.25)$$

and so THC entropy is a concave function of its probability distribution. Eq.(2.25) results from Jensen's inequality a concavity of $x^\alpha/(1-\alpha)$. In addition, by rule of l'Hospital we get that

$$\lim_{\alpha \rightarrow 1} \mathcal{S}_\alpha = \lim_{\alpha \rightarrow 1} \mathcal{I}_\alpha = \mathcal{H}. \quad (2.26)$$

Thus in the $\alpha \rightarrow 1$ limit THC entropy reduces to Shannon's entropy.

Perhaps the most distinguished feature of THC entropy is the so called pseudo-additivity [2,27]

$$\mathcal{S}_\alpha(\mathcal{A} \cap \mathcal{B}) = \mathcal{S}_\alpha(\mathcal{A}) + \mathcal{S}_\alpha(\mathcal{B}|\mathcal{A}) + (1-\alpha)\mathcal{S}_\alpha(\mathcal{A})\mathcal{S}_\alpha(\mathcal{B}|\mathcal{A}),$$

for two experiments \mathcal{A} and \mathcal{B} , $\mathcal{S}_\alpha(\mathcal{B}|\mathcal{A})$ represents here the conditional THC entropy. Remarkable, albeit not yet understood aspect of the pseudo-additivity is that in

the case of independent experiments THC entropy is not additive. Interested reader may find further discussion of THC entropy, for instance, in Ref. [28].

Now we turn to the problem of finding the connection between Rényi's and THC entropy. To this end we utilize the identity

$$\begin{aligned} \mathcal{I}_\alpha &= \frac{1}{(1-\alpha)} \log_2 [(1-\alpha)\mathcal{S}_\alpha + 1] \\ &= \frac{1}{k} \int_0^{\mathcal{S}_\alpha} dx \frac{1}{1+x(1-\alpha)}. \end{aligned} \quad (2.27)$$

Here $k = \ln 2$ is the scale factor. For $|(1-\alpha)\mathcal{S}_\alpha| < 1$ we may expand the integrand in (2.27). In such a case the (geometric) series is absolutely convergent and we can integrate it term by term:

$$\mathcal{I}_\alpha = \frac{1}{k} \mathcal{S}_\alpha - \frac{1}{2k} (1-\alpha) \mathcal{S}_\alpha^2 + \mathcal{O}[(1-\alpha)^2 \mathcal{S}_\alpha^3]. \quad (2.28)$$

So apart from an unimportant factor k (which just sets the scale for entropy units) we see that $\mathcal{I}_\alpha \approx \mathcal{S}_\alpha$, provided

$$|(1-\alpha)\mathcal{S}_\alpha| = \left| \sum_l^n (p_l)^\alpha - 1 \right| \ll 1. \quad (2.29)$$

It should be understood that the expansion (2.28) is not necessarily the expansion in $(1-\alpha)$. In fact, condition (2.29) may be fulfilled in numerous ways. Obviously, for $\alpha \approx 1$ the inequality (2.29) is trivially satisfied. This should be expected because both \mathcal{I}_α and \mathcal{S}_α tend to the same limit value at $\alpha \approx 1$. Thus the actual error estimate in this instance can be written as

$$\mathcal{I}_\alpha = \frac{1}{k} \mathcal{S}_\alpha + \mathcal{O}((\alpha-1)\mathcal{H}^2), \quad (2.30)$$

and so the true inaccuracy in dealing with \mathcal{S}_α and not \mathcal{I}_α is of order $(\alpha-1)$. There is, however, possible to pinpoint other very important classes of systems with $\alpha \not\approx 1$ still obeying (2.29). Clearly, various improved estimates can be devised if some additional assumptions are made about the system. One particularly important case which is pertinent to $\alpha < 1$ region, namely the case of large deviations will be briefly discussed now.

Systems with large deviations prove fruitful in many areas of physics and mathematics ranging from fluid dynamics and weather forecast to population breeding. To proceed we will appeal to Loève (or basic) inequality of probability theory [29]. Let X be an arbitrary random variable and let g be an even function on \mathbb{R} and non-decreasing on $[0, \infty)$. Then for $\forall a \geq 0$

$$\langle g(X) \rangle - g(a) \leq \sup g(X) P[|X| \geq a]. \quad (2.31)$$

Upon taking the distribution $\varrho(q) = \{(p_k)^q / \sum_k (p_k)^q\}$, $q \in [0, 1]$ and $g(x) = |x|^{\alpha-q}$, $\alpha \in [0, 1]$ we get from (2.31)

$$\langle |X|^{\alpha-q} \rangle_q - a^{\alpha-q} \leq \sup(|X|^{\alpha-q}) P[|X| \geq a]. \quad (2.32)$$

Here $\langle \dots \rangle_q$ is the mean with respect to $\varrho(q)$. We can now set $|X| = \mathcal{P} = \{p_k\}$ and fix q so to fulfill $\alpha > q$. Taking

$$\frac{1}{a} = \left(\sum_k^n (p_k)^q \right)^{1/(\alpha-q)} \equiv Z(q)^{1/(\alpha-q)}, \quad (2.33)$$

we obtain the probability theory variant of (2.29), namely

$$\begin{aligned} \sum_k^n (p_k)^\alpha - 1 &\leq \sup(\mathcal{P}^{\alpha-q}) P[\mathcal{P} \geq a] Z(q) \\ &\leq P[\mathcal{P} \geq a] Z(q). \end{aligned} \quad (2.34)$$

To proceed we realize that for $q \in [0, 1]$ we have $1 \leq Z(q) \leq n^{1-q}$ and hence

$$1 \geq a \geq \left(\frac{1}{n} \right)^{(1-q)/(\alpha-q)}. \quad (2.35)$$

Note particularly that $(1-q)/(\alpha-q) > 1$. Thus if for most of i 's the inequality $p_i \leq (1/n)^{(1-q)/(\alpha-q)}$ holds (rare events) then $P[\mathcal{P} \geq a]$ of (2.34) can be made arbitrarily small¹. Besides, because $Z(q)$ is bounded by n^{1-q} irrespective of a particular choice of \mathcal{P} and α we may use this freedom to fix RHS of (2.34) to be very small. So for example when most $p_i \approx 1/n^2$ then the choice $q = 1/2$ and $\alpha = 3/4$ assure that $Z(q) \leq \sqrt{n}$ while $P[\mathcal{P} \geq a] \approx 1/n$ and hence RHS of (2.34) is smaller than $1/\sqrt{n}$. It should be recognized that in this case the inequality (2.29) holds not because $\alpha \rightarrow 1$ but because n is large.

It is interesting to consider now the situation when $|(1-\alpha)\mathcal{S}_\alpha| > 1$. Such a case is undoubtedly more intriguing than the previous one as it represents a wider class of physically relevant situations. Let us start first with the situation $|(1-\alpha)\mathcal{S}_\alpha| \approx 1$. There are two cases of interest here. The case when $(1-\alpha)\mathcal{S}_\alpha \approx 1$ is the simpler one. Here $\alpha < 1$ due to positivity of \mathcal{S}_α and we may rewrite (2.27) as

$$\begin{aligned} k\mathcal{I}_\alpha &= \left(\int_0^{1/(1-\alpha)} + \int_{1/(1-\alpha)}^{\mathcal{S}_\alpha} \right) dx \frac{1}{1+x(1-\alpha)} \\ &= \frac{k}{(1-\alpha)} + \frac{\mathcal{S}_\alpha - 1/(1-\alpha)}{2} \\ &\quad + \mathcal{O} \left(\frac{[(1-\alpha)\mathcal{S}_\alpha - 1]^2}{(1-\alpha)} \right) \\ &\approx \frac{\mathcal{S}_\alpha}{2} + \frac{1}{(1-\alpha)} \left(k - \frac{1}{2} \right). \end{aligned} \quad (2.36)$$

¹Of course, due to normalization condition $\sum_i^n p_i = 1$, $P[\mathcal{P} \geq a]$ cannot be zero since there must be always a very small probability for large (i.e., $> 1/n$) p_i 's. Hence name *large deviations*.

On the other hand, the case when $(1-\alpha)\mathcal{S}_\alpha \approx -1$ is very important as it corresponds to the large α limit. Since for high α , \mathcal{S}_α asymptotically approaches $\zeta = [(p_k)_{\max}^\alpha - 1]/(1-\alpha)$ from above we can write

$$\begin{aligned} k\mathcal{I}_\alpha &= \left(\int_0^\zeta + \int_\zeta^{\mathcal{S}_\alpha} \right) dx \frac{1}{1+x(1-\alpha)} \\ &= \frac{\alpha \ln(p_k)_{\max}}{(1-\alpha)} + \frac{\mathcal{S}_\alpha(1-\alpha) + (1-(p_k)_{\max}^\alpha)}{(1-\alpha)(p_k)_{\max}^\alpha} \\ &\quad + \mathcal{O}([\mathcal{I}_\alpha + \log_2(p_k)_{\max}]^2) \\ &\approx \frac{\mathcal{S}_\alpha}{(p_k)_{\max}^\alpha} + \frac{(1+(p_k)_{\max}^\alpha[\alpha \ln(p_k)_{\max} - 1])}{(1-\alpha)(p_k)_{\max}^\alpha}. \end{aligned} \quad (2.37)$$

In both previous cases we have seen that the leading orders yielded a linear relationship between Rényi's and THC entropy. As already recognized by Schrödinger [30], statistical entropy is defined up to a linear transformation. This, in turn, one could view as a conceptual backing for THC entropy in the respective situations. One's pleasure is short-lived, however, when one starts to consider the case $(1-\alpha)\mathcal{S}_\alpha \gg 1$. This corresponds, for example, to the situation when $\alpha \rightarrow 0$. Writing (2.27) as

$$\begin{aligned} k\mathcal{I}_\alpha &= \left(\int_0^{1/(1-\alpha)} + \int_{1/(1-\alpha)}^{\mathcal{S}_\alpha} \right) dx \frac{1}{1+x(1-\alpha)} \\ &= \frac{k}{(1-\alpha)} + \sum_{n=0}^{\infty} (-1)^n \int_{1/(1-\alpha)}^{\mathcal{S}_\alpha} dx \left(\frac{1}{x(1-\alpha)} \right)^{n+1} \\ &\approx \frac{\ln(\mathcal{S}_\alpha(1-\alpha))}{(1-\alpha)} + \frac{1}{\mathcal{S}_\alpha(1-\alpha)^2}, \end{aligned} \quad (2.38)$$

we see that there is a logarithmic singularity at large \mathcal{S}_α . Hence, no linear mapping between RHC and Rényi's entropy exists in this region. One may thus expect that for $(1-\alpha)\mathcal{S}_\alpha \gg 1$ both entropies have qualitatively different behavior and the conceptual grounding for THC entropy must be sought out of the scope of information theory.

Let us add two more comments. It is often argued that concavity of THC entropy with respect to probability distribution makes it better suited, say, for thermodynamic considerations. It is, however, concavity with respect to extensive variables rather than probability distribution which ensures stability of thermodynamic equilibrium [14]. The first does not necessarily imply the second. Needless to say that there is no general concavity requirement for entropy in non-equilibrium systems. Secondly, from Eq.(2.27) we see that THC entropy and Rényi's entropy are monotonic functions of each other and, as a result, both must be maximized by the same probability distribution. However, while Rényi's entropy is additive, THC entropy is not, so that it appears that

the additivity property is not important for entropies required for maximization purposes.

III. RÉNYI'S ENTROPY OF CONTINUOUS PROBABILITY DISTRIBUTIONS

While in the previous section we dealt with the Rényi's entropy of discrete probability distributions we will now discuss the corresponding continuous counterpart. We shall see that in the latter case a host of new properties will emerge. As a byproduct we get a consistent extension of THC entropy for continuous distributions.

Let us first assume that $\mathcal{F}(x)$ is an arbitrary *continuous*, positive density function (PDF) defined, say, in the interval $[0, 1]$. By defining the integrated probability

$$p_{nk} = \int_{k/n}^{(k+1)/n} dx \mathcal{F}(x); \quad k = 0, 1, \dots, n-1,$$

we generate the discrete distribution $\mathcal{P}_n = \{p_{nk}\}$. It might be then shown [3,4] that

$$\begin{aligned} \mathcal{I}_\alpha(\mathcal{F}) &\equiv \lim_{n \rightarrow \infty} (\mathcal{I}_\alpha(\mathcal{P}_n) - \log_2 n) \\ &= \frac{1}{1-\alpha} \log_2 \left(\int_0^1 dx \mathcal{F}^\alpha(x) \right), \end{aligned} \quad (3.1)$$

provided that $\int_0^1 dx \mathcal{F}^\alpha(x)$ exists². Here $\log_2 n$ must be subtracted to ensure a correct measure in the integral. Defining the uniform distribution $\mathcal{E}_n = \{\frac{1}{n}, \dots, \frac{1}{n}\}$ then $\log_2 n = \mathcal{I}_\alpha(\mathcal{E}_n)$. From this we may interpret $-\mathcal{I}_\alpha(\mathcal{F}) \sim \mathcal{I}_\alpha(\mathcal{E}_n) - \mathcal{I}_\alpha(\mathcal{P}_n)$ as the gain of information obtained by replacing the uniform distribution \mathcal{E}_n (having maximal uncertainty) by distribution \mathcal{P}_n or, in other words, $-\mathcal{I}_\alpha(\mathcal{F})$ represents the decrease of uncertainty when \mathcal{E}_n is replaced by \mathcal{P}_n . In the case of Shannon's entropy the quantity $-\mathcal{H}(\mathcal{F})$ is usually called the informative content or "negentropy" and states how much uncertainty is still left unresolved after a measurement (for discussion see e.g., [33,34]).

Relation (3.1) can be viewed as a *renormalized* Rényi's information content. This may be understood from the asymptotic expansion of $\mathcal{I}_\alpha(\mathcal{P}_n)$, namely

$$\mathcal{I}_\alpha(\mathcal{P}_n) = \text{divergent in } n + \text{finite} + o(1), \quad (3.2)$$

the $o(1)$ symbol means that the residual error tends to 0 for $n \rightarrow \infty$. The *finite* part ($= \mathcal{I}_\alpha(\mathcal{F})$) is fixed by requirement (or by renormalization prescription) that it should fulfill the postulate of additivity in order to be

identifiable with an information measure. Incidentally, the latter uniquely identifies the divergent part as $\log_2 n$. The above renormalization procedure is somehow analogous to that in quantum field theory where one renormalizes energy by subtracting the ground state contribution. It should be, however, noted that the information $\log_2 n$ is usually greater than $\mathcal{I}_\alpha(\mathcal{P}_{nk})$ and consequently $\mathcal{I}_\alpha(\mathcal{F})$ is not positive. The former should be contrasted with the discrete case where \mathcal{I}_α is by construction non-negative.

Extension of (3.1) into d -dimensional situations is straightforward. Having a d -dimensional random variable (i.e., experiment) $\mathcal{A}^{(d)}$ we can discretize it in the following way; $\mathcal{A}_n^{(d)} = \left(\frac{[n\mathcal{A}_1]}{n}, \frac{[n\mathcal{A}_2]}{n}, \dots, \frac{[n\mathcal{A}_d]}{n} \right)$ where [...] denotes integral part. This divides the d -dimensional volume V of the outcome (or sample) space into boxes labelled by an index k which runs from 1 up to $[Vn^d]$. The size of the k th box is $l = 1/n$ and its probability distribution $\mathcal{P}_n^{(d)} = \{p_{nk}^{(d)}\}$ is generated via prescription

$$p_{nk}^{(d)} = \int_{\text{kth box}} d^d \mathbf{x} \mathcal{F}(\mathbf{x}); \quad k = 1, 2, \dots, [Vn^d].$$

It can be shown then (see e.g., [3] and Appendix D) that

$$\begin{aligned} \mathcal{I}_\alpha^{(d)}(\mathcal{F}) &\equiv \lim_{n \rightarrow \infty} (\mathcal{I}_\alpha(\mathcal{P}_n^{(d)}) - d \log_2 n) \\ &= \frac{1}{(1-\alpha)} \log_2 \left(\int_V d^d \mathbf{x} \mathcal{F}^\alpha(\mathbf{x}) \right), \end{aligned} \quad (3.3)$$

provided that $\int_V d^d \mathbf{x} \mathcal{F}^\alpha(\mathbf{x})$ exists.

Question now stands whether we get unique $\mathcal{I}_\alpha^{(d)}(\mathcal{F})$ by mimicking the previous recipe, i.e., performing the asymptotic expansion of $\mathcal{I}_\alpha(\mathcal{P}_n^{(d)})$ and pinpointing the correct finite part by the renormalization condition - additivity of information. In the non-unit volume, however, one more fixing condition is required. To see that we define the uniform distribution $\mathcal{E}_n^{(d)} = \left\{ \frac{1}{Vn^d}, \dots, \frac{1}{Vn^d} \right\}$ with $V_n \equiv \frac{[Vn]}{n} \xrightarrow{n \rightarrow \infty} V$. Rényi's entropy then reads

$$\mathcal{I}_\alpha(\mathcal{E}_n^{(d)}) = \log_2 V_n + d \log_2 n,$$

and so

$$\begin{aligned} \tilde{\mathcal{I}}_\alpha^{(d)}(\mathcal{F}) &\equiv \lim_{n \rightarrow \infty} (\mathcal{I}_\alpha(\mathcal{P}_n^{(d)}) - \mathcal{I}_\alpha(\mathcal{E}_n^{(d)})) \\ &= \frac{1}{(1-\alpha)} \log_2 \left(\frac{\int_V d^d \mathbf{x} \mathcal{F}^\alpha(\mathbf{x})}{\int_V d^d \mathbf{x} 1/V^\alpha} \right). \end{aligned} \quad (3.4)$$

Alike in (3.3) the RHS of (3.4) represents the finite part in the asymptotic expansion of $\mathcal{I}_\alpha(\mathcal{P}_n^{(d)})$, the part which fulfils the additivity of information condition. To ensure the uniqueness of Rényi entropy in the case of continuous distributions we must, in addition, fix the value of the finite part at $\mathcal{F} = (1/V)$. It is then matter of taste and/or a particular problem at hand which convention should be used. In this paper we will use the renormalization prescription where $\mathcal{I}_\alpha^{(d)}(1/V)|_{\text{finite}} = \log_2 V$ (i.e.,

²For $0 < \alpha < 1$ this is always the case as $\sum_k (p_{nk})^\alpha \leq n^{1-\alpha} \Rightarrow \int_0^1 dx \mathcal{F}^\alpha(x) \leq 1$.

the one which implies Eq.(3.3)). The latter merely means that we define Rényi's entropy with PDF \mathcal{F} as

$$\mathcal{I}_\alpha^{(d)}(\mathcal{F}) \equiv \lim_{n \rightarrow \infty} (\mathcal{I}_\alpha(\mathcal{P}_n^{(d)}) - \mathcal{I}_\alpha(\mathcal{E}_n^{(d)})|_{V=1}). \quad (3.5)$$

In Section IV we generalize results (3.4) and (3.5) into fractal and multifractal systems. A comment is in order. It may be shown (see Appendix E) that the form (3.4) is, in fact, a better candidate for the information measure than (3.3) as it is an invariant under a transformation of $\mathcal{A}^{(d)}$. However, difference between (3.3) and (3.4) is often only a constant which ensures that for the questions we address here it is quite adequate to use the simpler form (3.3). It should be, however, clear that there are system of physical interest where the ground-state entropy plays a central rôle (e.g., frustrated spin systems or quantum liquids). In such cases the form (3.4) is obligatory.

Let us now examine the implications of (3.1)–(3.4) for THC entropy with continuous distributions. For this we will use the convention introduced before Eq.(3.3). Firstly, from (2.27) and (3.3) follows that $[\mathcal{I}_\alpha(\mathcal{P}_n) - d \log_2 n]$ is finite at large n (provided $\int_V d^d \mathbf{x} \mathcal{F}^\alpha(\mathbf{x})$ exists) and so

$$\frac{(1-\alpha)\mathcal{S}_\alpha(\mathcal{P}_n) + 1}{n^{d(1-\alpha)}} = \int_V d^d \mathbf{x} \mathcal{F}^\alpha(\mathbf{x}) + o(1). \quad (3.6)$$

In order to obtain the correct THC entropy with PDF \mathcal{F} it is conceptually simplest to follow the same route as before, i.e., asymptotically expand $\mathcal{S}_\alpha(\mathcal{P}_n)/n^{d(1-\alpha)}$ and look for the finite part which conforms to certain renormalization prescription³. Unlike the Rényi entropy case we do not have now any first principle renormalization prescription (*à la* additivity of information) which we could impose. As a matter of fact, one could be tempted to use the THC pseudo-additivity condition to isolate the proper finite part in the $\mathcal{S}_\alpha(\mathcal{P}_n)/n^{d(1-\alpha)}$ expansion, but such a renormalization condition would be clearly *ad hoc* as there is no a priori reason to assume that the non-extensivity condition obeys the same prescription in the continuous case. It is fairly safer to follow the analogy with Eqs.(3.4) and (3.5) demanding, for instance, the consistency for α 's in the complex vicinity of $\alpha = 1$ (i.e., values at which Rényi and THC entropies coincide). If the consistency is reached then the validity of the result can be analytically continued to the whole domain of analyticity of \mathcal{S}_α - so particularly to $\alpha \in \mathbb{R}^+$.

Using the asymptotic expansions:

³It is indeed $\mathcal{S}_\alpha(\mathcal{P}_n)/n^{d(1-\alpha)}$ rather than $\mathcal{S}_\alpha(\mathcal{P}_n)$ which should be asymptotically expanded. For instance, for $0 < \alpha < 1$ the asymptotic expansion of $\mathcal{S}_\alpha(\mathcal{P}_n)$ would be $o(1)$ and so the corresponding large n limit would be trivial. It is not difficult to see that it is only the fraction $\mathcal{S}_\alpha(\mathcal{P}_n)/n^{d(1-\alpha)}$ which has a sensible meaning in the large n limit.

$$\begin{aligned} \frac{\mathcal{S}_\alpha(\mathcal{P}_n^{(d)})}{n^{d(1-\alpha)}} &= -\frac{1}{(1-\alpha)n^{d(1-\alpha)}} \\ &+ \frac{1}{(1-\alpha)} \int_V d^d \mathbf{x} \mathcal{F}^\alpha(\mathbf{x}) + o(1), \\ \frac{\mathcal{S}_\alpha(\mathcal{E}_n^{(d)})}{n^{d(1-\alpha)}} &= -\frac{1}{(1-\alpha)n^{d(1-\alpha)}} \\ &+ \frac{1}{(1-\alpha)} \int_V d^d \mathbf{x} 1/V^\alpha + o(1), \end{aligned} \quad (3.7)$$

we may immediately write

$$\begin{aligned} \tilde{\mathcal{S}}_\alpha^{(d)}(\mathcal{F}) &\equiv \lim_{n \rightarrow \infty} \left(\frac{\mathcal{S}_\alpha(\mathcal{P}_n^{(d)})}{n^{d(1-\alpha)}} - \frac{\mathcal{S}_\alpha(\mathcal{E}_n^{(d)})}{n^{d(1-\alpha)}} \right) \\ &= \frac{1}{(1-\alpha)} \left(\int_V d^d \mathbf{x} \mathcal{F}^\alpha(\mathbf{x}) - 1 \right) \\ &- \frac{1}{(1-\alpha)} \left(\int_V d^d \mathbf{x} 1/V^\alpha - 1 \right), \\ \mathcal{S}_\alpha^{(d)}(\mathcal{F}) &\equiv \lim_{n \rightarrow \infty} \left(\frac{\mathcal{S}_\alpha(\mathcal{P}_n^{(d)})}{n^{d(1-\alpha)}} - \frac{\mathcal{S}_\alpha(\mathcal{E}_n^{(d)})|_{V=1}}{n^{d(1-\alpha)}} \right) \\ &= \frac{1}{(1-\alpha)} \left(\int_V d^d \mathbf{x} \mathcal{F}^\alpha(\mathbf{x}) - 1 \right). \end{aligned} \quad (3.8)$$

It is not difficult to check that for $|\alpha| \in [1-\epsilon, 1+\epsilon]$, $\epsilon \ll 1$, (3.8) is consistent with (3.4) and (3.5).

Let us note at the end that from the asymptotic expansion of $\mathcal{I}_\alpha(\mathcal{P}_n^{(d)})$ i.e., from

$$\mathcal{I}_\alpha(\mathcal{P}_n^{(d)}) = d \log_2 n + \mathcal{I}_\alpha^{(d)}(\mathcal{F}) + o(1), \quad (3.9)$$

we find, in return, that the dimension d is identified with

$$d(\alpha) = \lim_{n \rightarrow \infty} \frac{\mathcal{I}_\alpha(\mathcal{P}_n^{(d)})}{\log_2 n}. \quad (3.10)$$

For simple metric (outcome) spaces (like \mathbb{R}^d) we will prove in the following section that $d(\alpha) = d$ for all α and it coincides with the usual topological dimension. This situation is however not generic. In the next section we shall see what modifications should be done when (multi)fractal systems are in question.

IV. RÉNYI'S PARAMETER AND (MULTI)FRACTAL DIMENSION

Fractals, objects with a generally non-integer dimension exhibiting the scaling property and property of self-similarity have had a significant impact not only on mathematics but also on such distinctive fields as physical chemistry, astrophysics, physiology, and fluid mechanics. The key characteristic of fractals is fractal dimension which is defined as follows: Consider a set M embedded in a d -dimensional space. Let us cover the set with a mesh of d -dimensional cubes of size l^d and let $N_l(M)$ is a minimal number of the cubes needed for the covering.

The fractal dimension (or similarity dimension) of M is then defined as [35,36]

$$D = -\lim_{l \rightarrow 0} \frac{\ln N_l(M)}{\ln l}. \quad (4.1)$$

In most cases of interest the fractal dimension (4.1) coincides with the Hausdorff–Besicovich fractal dimension used by Mandelbrot [35].

Multifractals, on the other hand, are related to the study of a distribution of physical or other quantities on a generic support (be it or not fractal) and thus provide a move from the geometry of sets as such to geometric properties of distributions. An intuitive picture about an inner structure of multifractals is obtained by introducing the $f(a)$ spectrum [5,37]. To elucidate the latter let us suppose that over some support (usually a subset of a metric space) is distributed a probability of a certain phenomenon, be it e.g., probability of electric charge, magnetic momenta, hydrodynamic vorticity or mass. If we cover the support with boxes of size l and denote the integrated probability in the i th box as p_i , we may define the local scaling exponent a_i by

$$p_i(l) \sim l^{a_i}. \quad (4.2)$$

where a_i is called the Lipschitz–Hölder exponent. Here and throughout the symbol \sim indicates an asymptotic relation, e.g., (4.2) should read:

$$a_i = \lim_{l \rightarrow 0} \frac{\ln p_i(l)}{\ln l}.$$

The proportionality constant (say $c(a_i)$) in (4.2) can be weakly dependent on l . By “weakly” we mean that

$$\lim_{l \rightarrow 0} \frac{\ln c(a_i, l)}{\ln l} = 0.$$

Note that PDF of each of small pieces is

$$\rho_i = \frac{p_i}{l^d} \sim l^{a_i - d}, \quad (4.3)$$

and so a_i controls the singularity of ρ_i . Inasmuch a_i is also known as the singularity exponent.

Counting number of boxes $dN(a)$ where p_i has singularity exponent between a and $a + da$, then $f(a)$ defines the fractal dimension of the set of boxes with the singularity exponent a by

$$dN(a) \sim l^{-f(a)} da. \quad (4.4)$$

Here $f(a)$ is called singularity spectrum. Multifractal can be then viewed as the ensemble of intertwined (uni)fractals each with its own fractal dimension $f(a)$. So $f(a)$ describes how densely the subsystems with the singularity exponent a are distributed. It should be noted that power law behaviors (4.2) and (4.4) are the fundamental assumptions of the multifractal analysis.

The convenient way how to keep track with p_i ’s is to examine the scaling of the corresponding moments. For this purpose one can define a “partition function” as

$$Z(q) = \sum_i p_i^q = \int da \, n(a) l^{-f(a)} l^{qa}, \quad (4.5)$$

($n(a)$ is (weakly l dependent) proportionality function having its origin in relations (4.2) and (4.4)). In the small l case the asymptotic behavior of the partition function can be evaluated by the method of steepest descents. As a result we get the scaling

$$Z(q) \sim l^\tau, \quad (4.6)$$

with

$$\begin{aligned} \tau(q) &\equiv \min_a (qa - f(a)) = qa_0(q) - f(a_0(q)), \\ &\Rightarrow f'(a_0(q)) = q \quad \text{and} \quad a_0(q) = \tau'(q). \end{aligned} \quad (4.7)$$

These are precisely the Legendre transform relations. Scaling function $\tau(q)$ is called correlation exponent or mass exponent of the q th order. So for the purpose of multifractal description we may use either of the conjugated couples $f(a_0)$, a_0 or $\tau(q)$, q . For the future reference we will need to know that $\tau(0) = -D$ and $\tau(1) = 0$ (see e.g., [35]). Let us finally stress that if not stated otherwise, we will often “abuse” notation and write simply a instead of a_0 .

A. Generalization of Eqs.(3.4) and (3.5) to fractal sample spaces and multifractals

With the definitions of (multi)fractal dimensions at hand we may now generalize Eqs.(3.4) and (3.5). Let us assume first that we have a fractal support M on which is defined a continuous PDF $\mathcal{F}(x)$. Following the renormalization prescription of Section III we know that in order to obtain the renormalized Renyi’s entropy we have to know $\mathcal{I}_\alpha(\mathcal{E}_n)$. This can be done by realizing that the uniform distribution is now $\mathcal{E}_n = \left\{ \frac{1}{N_l}, \dots, \frac{1}{N_l} \right\}$. Here N_l is the minimal covering (with cubes of size l^d) of the fractal set in question and $n = 1/l$. Due to scaling law (4.1) the (pre)fractal volume $V_l = N_l l^D$ converges to the actual (finite) fractal volume V in the $l \rightarrow 0$ limit. As a result $\mathcal{E}_n = \left\{ \frac{l^D}{V_l}, \dots, \frac{l^D}{V_l} \right\}$, and hence

$$\mathcal{I}_\alpha(\mathcal{E}_n) = \log_2 V_l - D \log_2 l. \quad (4.8)$$

In the $n \rightarrow \infty$ (i.e., $l \rightarrow 0$) limit we prove in Appendix D that either

$$\begin{aligned} \tilde{\mathcal{I}}_\alpha(\mathcal{F}) &\equiv \lim_{n \rightarrow \infty} (\mathcal{I}_\alpha(\mathcal{P}_n) - \mathcal{I}_\alpha(\mathcal{E}_n)) \\ &= \frac{1}{(1 - \alpha)} \log_2 \left(\frac{\int_M d\mu \mathcal{F}^\alpha(\mathbf{x})}{\int_M d\mu 1/V^\alpha} \right), \end{aligned} \quad (4.9)$$

or

$$\begin{aligned}\mathcal{I}_\alpha(\mathcal{F}) &\equiv \lim_{n \rightarrow \infty} (\mathcal{I}_\alpha(\mathcal{P}_n) - \mathcal{I}_\alpha(\mathcal{E}_n)|_{V=1}) \\ &= \lim_{n \rightarrow \infty} (\mathcal{I}_\alpha(\mathcal{P}_n) + D \log_2 l) \\ &= \frac{1}{(1-\alpha)} \log_2 \left(\int_M d\mu \mathcal{F}^\alpha(\mathbf{x}) \right),\end{aligned}\quad (4.10)$$

in conformity with the chosen renormalization prescription. The measure μ is the Hausdorff measure. Note that the RHS's of (4.9) and (4.10) are finite provided the integral $\int_M d\mu \mathcal{F}^\alpha(\mathbf{x})$ exists. From (4.10) the asymptotic expansion (3.9) for $\mathcal{I}_\alpha(\mathcal{P}_n)$ reads

$$\mathcal{I}_\alpha(\mathcal{P}_n) = D \log_2 n + \mathcal{I}_\alpha(\mathcal{F}) + o(1). \quad (4.11)$$

This means that $d(\alpha)$ defined in (3.10) boils down to

$$d(\alpha) = \lim_{n \rightarrow \infty} \frac{\mathcal{I}_\alpha(\mathcal{P}_n)}{\log_2 n} = D, \quad \text{for } \forall \alpha. \quad (4.12)$$

We remark that the information measure $D \log_2 n$ appearing in (5.47) and (4.10) is nothing but an information-theoretical analogue of the Boltzmann entropy: $S = k_B \ln W$ (k_B is the Boltzmann constant and W is the number of accessible microstates). This is so because both $\mathcal{I}_\alpha(\mathcal{E}_n)$ ($= \mathcal{H}(\mathcal{E}_n)$ for $\forall \alpha$) and the Boltzmann entropy S describe systems where all possible outcomes (or accessible microstates) have assigned equal probabilities (constant PDF). Thus $\mathcal{I}_\alpha(\mathcal{E}_n)$ alike S are both maximal attainable entropies compatible with a given set of all possible outcomes (or accessible microstates).

Foregoing analysis can be also utilized to multifractals. In fact, by employing the multifractal measure [36]

$$\mu_{\mathcal{P}}^{(\alpha)}(d; l) = \sum_{k \text{th box}} \frac{p_{nk}^\alpha}{l^d} \xrightarrow{l \rightarrow 0} \begin{cases} 0 & \text{if } d < \tau(\alpha) \\ \infty & \text{if } d > \tau(\alpha) \end{cases}, \quad (4.13)$$

we prove in Appendix F that

$$\begin{aligned}\mathcal{I}_\alpha(\mu_{\mathcal{P}}) &\equiv \lim_{l \rightarrow 0} (\mathcal{I}_\alpha(\mathcal{P}_n) - \mathcal{I}_\alpha(\mathcal{E}_n)|_{V=1}) \\ &= \lim_{l \rightarrow 0} \left(\mathcal{I}_\alpha(\mathcal{P}_n) + \frac{\tau(\alpha)}{(\alpha-1)} \log_2 l \right) \\ &= \frac{1}{(1-\alpha)} \log_2 \left(\int_a d\mu_{\mathcal{P}}^{(\alpha)}(a) \right).\end{aligned}\quad (4.14)$$

Eq.(4.14) implies the asymptotic expansion

$$\mathcal{I}_\alpha(\mathcal{P}_n) = \frac{\tau(\alpha)}{(\alpha-1)} \log_2 n + \mathcal{I}_\alpha(\mu_{\mathcal{P}}) + o(1). \quad (4.15)$$

Consequently we note that $d(\alpha)$ of (3.10) reads

$$d(\alpha) = \lim_{n \rightarrow \infty} \frac{\mathcal{I}_\alpha(\mathcal{P}_n)}{\log_2 n} = \frac{\tau(\alpha)}{(\alpha-1)}. \quad (4.16)$$

Unlike in fractal sample spaces, in multifractals $d(\alpha)$ depends on α . Note that in the case of smooth PDF's

the integrated probability $p_i(l)$ scales as $l^{f(a)}$ and so we have a unifractal characterized by a single dimension $a = f(a) \equiv D$. This implies that $\tau/(\alpha-1) = D$ and hence for smooth PDF's we naturally recover the result (4.12). It should be emphasized that when the outcome space is a simple metric space (like \mathbb{R}^d) then it is known that the fractal dimension D coincides with the usual topological dimension [35,36] and so, for instance, $D = d$ in the case of \mathbb{R}^d .

B. Generalized dimensions and reconstruction theorem

After this brief intermezzo we now turn back to the question whether there is any connection of Rényi's entropy with (multi)fractal systems. At present it seems to us that there are at least two such connections. The first, more formal connection, is associated with the so called generalized dimensions of the q th order defined as:

$$\mathcal{D}_q \equiv \lim_{l \rightarrow 0} \left(\frac{1}{(q-1)} \frac{\ln Z_q}{\ln l} \right) = \frac{\tau(q)}{(q-1)}. \quad (4.17)$$

In passing the reader should notice that \mathcal{D}_q is nothing but $d(\alpha = q)$ introduced in (4.16). A complete knowledge of the collection of generalized dimensions \mathcal{D}_q is equivalent to a complete physical characterization of the fractal [39]. It should be noted in this connection that the fractal dimension, the information dimension and the correlation dimension (all frequently used in the deterministic chaotic systems [40]) are, respectively \mathcal{D}_0 , \mathcal{D}_1 and \mathcal{D}_1 . In fact, all \mathcal{D}_q are necessary to describe uniquely general fractals e.g., strange attractors [39]. This is analogous to statistical physics where one needs all cumulants to get the full density matrix. Mathematically this corresponds to Hausdorff's moment problem [41].

While the proof in [39] is based on a rather complicated self-similarity argumentation we can understand the core of this assertion using a different angle of view. In fact, employing the information theory we will show that the assumption of a self-similarity is not really fundamental and that the conclusion of [39] has more general applicability. For this purpose let us define the information-distribution function of \mathcal{P} (see e.g., [4]) as

$$\mathcal{F}_{\mathcal{P}}(x) = \sum_{-\log_2 p_k < x} p_k. \quad (4.18)$$

The latter represents the total probability carried out by events with information contents $\mathcal{I}_k = -\log_2 p_k < x$. Note also that for $x < 0$ the sum in (4.18) is empty and so $\mathcal{F}_{\mathcal{P}}(x) = 0$. Realizing that

$$2^{(1-\alpha)x} d\mathcal{F}_{\mathcal{P}}(x) \approx \sum_{x \leq \mathcal{I}_k < x+dx} 2^{(1-\alpha)\mathcal{I}_k} p_k = \sum_{x \leq \mathcal{I}_k < x+dx} p_k^\alpha,$$

we may write

$$\mathcal{I}_\alpha(\mathcal{P}) = \frac{1}{(1-\alpha)} \log_2 \left(\int_{x=0}^{\infty} 2^{(1-\alpha)x} d\mathcal{F}_\mathcal{P}(x) \right). \quad (4.19)$$

The former integral should be understood in the Stieltjes sense ($\mathcal{F}_\mathcal{P}(x)$ is generally discontinuous). Taking the inverse Laplace–Stieltjes transform of (4.19) we obtain

$$\begin{aligned} \mathcal{F}_\mathcal{P}(x) &= \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp \frac{e^{px} e^{-p\mathcal{I}_\alpha(\mathcal{P})}}{p} \\ &= \sum_l \frac{p_l}{2\pi i} \int_{-i\infty+0_+}^{i\infty+0_+} dp \frac{e^{p(x+\log_2 p_l)}}{p}, \end{aligned} \quad (4.20)$$

with $p = (\alpha - 1) \ln 2$. The constant σ is dictated by requirements that it should be positive and that all singularities of $\frac{e^{-p\mathcal{I}_\alpha}}{p}$ should lie to the left of the vertical line $\Re(p) = \sigma$ in the complex p -plane. As $e^{-p\mathcal{I}_\alpha}$ is basically $\sum_k p_k^\alpha$ it means that $\frac{e^{-p\mathcal{I}_\alpha}}{p}$ is analytic on the half-plane $\{p \mid \Re(p) > 0\}$. As a result we may choose $\sigma = 0_+$. For $(x + \log_2 p_k) < 0$ we may close the contour by a semicircle in the right half of the plane. In this region integrand is analytic and so $\mathcal{F}_\mathcal{P}(x) = 0$ as it should be. For $(x + \log_2 p_k) > 0$, the semicircle must be placed in the left half plane, which yields then correct $\mathcal{F}_\mathcal{P}(x)$ of Eq.(4.18).

Disadvantage of the inverse formula (4.20) is that p (and so α) gets its values from \mathbb{C} , or more specifically, one needs (at best) all complex p 's belonging to the small circle around $p = 0$ to reconstruct the underlying distribution. It is however clear that in order to determine how many α 's are really needed to fully reconstruct \mathcal{P} one must resort to the real inverse Laplace transform instead. Such a reversal indeed exists and is provided by, the so called, Widder–Stieltjes inverse formula [41]:

$$\mathcal{F}_\mathcal{P}(x) \approx \sum_{n=0}^{\Lambda} \frac{\left(-\frac{\Lambda}{x}\right)^n}{n!} \left[\exp\left(-\frac{\Lambda}{x} \mathcal{I}_{(\Lambda/\ln(2)x+1)}\right) \right]^{(n)},$$

or (after setting $\frac{\Lambda}{x} = z$)

$$\mathcal{F}_\mathcal{P}\left(\frac{\Lambda}{z}\right) \approx \sum_{n=0}^{\Lambda} \frac{(-z)^n}{n!} \left[\exp\left(-z \mathcal{I}_{(z/\ln(2)+1)}\right) \right]^{(n)}, \quad (4.21)$$

here Λ is a regulator which has to be set to $+\infty$ at the end of calculations. It is important to recognize that the RHS of (4.21) depends on all $\alpha \in [1, \infty)$. Other, more intuitive, proof of the same fact is provided in Appendix G. In addition, in Appendix H we show that a similar “reconstruction” theorem holds also for THS entropy \mathcal{S}_α .

As a result, when working with \mathcal{I}_α of different orders we receive more information than restricting our consideration to only one α . In this connection it is illuminating to rewrite the complex integral in (4.20) as

$$\begin{aligned} \int_{-i\infty+0_+}^{i\infty+0_+} dp \frac{e^{p(x+\log_2 p_k)}}{p} \\ = \text{PP} \int_{-\infty}^{\infty} dp \frac{e^{ip(x+\log_2 p_k)}}{p} + i\pi. \end{aligned} \quad (4.22)$$

Here PP stands for the principal part (associated to the pole at $p = 0$). The term $i\pi$ is the sole contribution from $p = 0$ (i.e., $\alpha = 1$), while PP(...) part corresponds to the contribution from the (imaginary axis) neighborhood of $p = 0$. In the case when $(x + \log_2 p_k) > 0$ then $\text{PP}(\dots) = i\pi$ and when $(x + \log_2 p_k) < 0$ then $\text{PP}(\dots) = -i\pi$, so the $\alpha = 1$ contribution has precisely 50% dominance. It should be also realized that PP(...) is ruled for most p_k 's by p 's from the close proximity of $p = 0$. In fact,

$$\begin{aligned} \text{PP} \int_{-\infty}^{\infty} dp \frac{e^{ip(x+\log_2 p_k)}}{p} \\ = \text{PP} \int_{-\delta}^{\delta} dp \frac{e^{ip(x+\log_2 p_k)}}{p} - 2i \text{si}(\delta y) \\ \approx \text{PP} \int_{-\delta}^{\delta} dp \frac{e^{ip(x+\log_2 p_k)}}{p} \\ + 2i\varepsilon(y) (\pi/2 - \delta|y| + \mathcal{O}((\delta|y|)^3)), \end{aligned} \quad (4.23)$$

with δ being the δ -neighborhood of $p = 0$, $\text{si}(x)$ being the sine integral and $y = (x + \log_2 p_k)$. Hence we see that when the outcome space is a discrete set we need generally all \mathcal{I}_α 's with $\alpha \in [1, \infty)$ to determine \mathcal{P} albeit the most dominant contribution comes from the relatively small neighborhood of $\mathcal{I}_1 = \mathcal{H}$. The latter statement is the discrete-space variant of the conclusion in [39].

Let us now briefly comment on the reconstruction theorem for the cases when the outcome space is a d -dimensional subset of \mathbb{R}^d . By covering the subset with the mesh of d -dimensional cubes of size $l^d = 1/n^d$ we obtain similarly as in Section III the integrated distributions $\mathcal{P}_n = \{p_{nk}\}$ and $\mathcal{E}_n = \{\mathcal{E}_{nk}\}$. The corresponding information–distribution function now reads

$$\begin{aligned} \mathcal{F}_{\mathcal{P}_n/\mathcal{E}_n}(x) &= \sum_{-\log_2(p_{nk}/\mathcal{E}_{nk}) < x} (p_{nk}/\mathcal{E}_{nk}) / \sum_k (p_{nk}/\mathcal{E}_{nk}) \\ &= \sum_{-\log_2(p_{nk}/\mathcal{E}_{nk}) < x} (p_{nk}/\mathcal{E}_{nk}) \frac{1}{Vn^d}. \end{aligned} \quad (4.24)$$

This implies (for $V = 1$) that

$$\int_{x=-d\log_2 n}^{\infty} 2^{(1-\alpha)x} d\mathcal{F}_{\mathcal{P}_n/\mathcal{E}_n}(x) = \frac{\sum_k p_{nk}^\alpha}{\sum_k \mathcal{E}_{nk}^\alpha},$$

and so in accord with (3.3)

$$\begin{aligned} \mathcal{I}_\alpha^{(n)}(\mathcal{F}) &= \frac{1}{(1-\alpha)} \log_2 \left(\int_{x=-d\log_2 n}^{\infty} 2^{(1-\alpha)x} d\mathcal{F}_{\mathcal{P}_n/\mathcal{E}_n}(x) \right), \\ \mathcal{I}_\alpha(\mathcal{F}) &= \lim_{n \rightarrow \infty} \mathcal{I}_\alpha^{(n)}(\mathcal{F}). \end{aligned} \quad (4.25)$$

Using the Widder–Stieltjes inverse formula we may recreate $\mathcal{F}_{\mathcal{P}_n/\varepsilon_n}(x)$ (and hence \mathcal{F}) in terms of $\mathcal{I}_\alpha^{(n)}(\mathcal{F})$'s. But the important moral here is that in the continuous limit (large n) $x \in (-\infty, \infty)$ and so $\alpha \in (-\infty, \infty)$. Unlike in discrete sample spaces, all \mathcal{I}_α , including those with $\alpha < 1$, are needed now to pinpoint the underlying PDF.

It should be born in mind that from a purely mathematical point of view the reconstruction procedure presented here is by no means the proof which extends easily to (multi)fractal systems - there is now obvious analogue of the Widder–Stieltjes inverse formula there. It should be rather taken as an indication that in general systems all \mathcal{I}_α with $\alpha \in (-\infty, \infty)$ are needed to determine uniquely the probability distribution. This is basically a weak version of the celebrated *moment problem of Hausdorff* [41]. The latter resonates with the finding that for deterministic chaotic systems the multifractal scaling function $\tau(q)$ often exists even for negative values of q . In those cases the partition function (4.5) is dominated by very small values of p_i . Hence one may be skeptical about the real existence of such a negative- q scaling behavior since the latter can be easily disrupted by fluctuations. In fact, if we explore the stability of Rényi's entropy for negative α by adding a small imaginary part into α we obtain Fig.1.

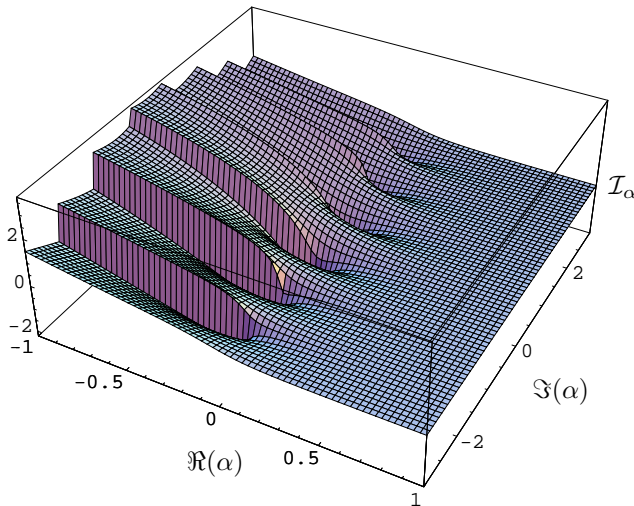


FIG. 1. A plot of Rényi's entropy $\mathcal{I}_\alpha(\mathcal{P})$ for 2 dimensional $\mathcal{P} = (p_1, p_2) = (p, 1 - p)$. We choose $p = 0.01$.

As p goes closer to zero there is a violent proliferation of branch cuts in the left half of the complex α -plane. So information conveyed by Rényi's entropy with negative α starts to be highly unreliable. Because Rényi's entropy is connected with the generalized dimensions via relation (4.17) such a breakdown of scaling for negative q 's (and hence α 's) should be inevitable in various deterministic chaotic systems. This is indeed the case, see e.g., [46].

The former reasonings may, to a certain extent, vindicate the use of $\alpha \geq 0$ in usual information theory. The bound $\alpha \geq 0$ can be hence merely understood as a reliability bound imposed on the conveyed information.

C. Thermodynamic formalism and MaxEnt

The second connection which we intend to advocate and progress here is the connection with the maximal entropy principle (MaxEnt). We will show that from the MaxEnt point of view, extremizing Shannon's entropy on (multi)fractals is equivalent to extremizing directly Rényi's entropy without invoking the (multi)fractal structure explicitly. An explicit illustration of this point on the network of cosmic strings will be given elsewhere.

Consider a support paved with boxes of size l and let the integrated probability in the k th box is denoted as p_k . Shannon's entropy of such a process is then

$$\mathcal{I} = - \sum_k p_k(l) \log_2 p_k(l)$$

The important observation of the multifractal theory is that for $q = 1$

$$a(1) = \frac{d\tau(1)}{dq} = \lim_{l \rightarrow 0} \frac{\sum_k p_k(l) \log_2 p_k(l)}{\log_2 l}. \quad (4.26)$$

It can be shown that the number $a(1) = f(a(1))$ describes the Hausdorff–Besicovich dimension of the set on which the probability is concentrated (see e.g., [36]). This means that the probability distribution \mathcal{P}_n is cumulated on the l -mesh cubes with $p_k(l) \sim l^{a(1)}$. In fact, the relative probability of the complement set approaches zero in the $l \rightarrow 0$ limit [36]. This statement goes also under the name *Billingsley theorem* [42] or *curdling* [35]. The corresponding subset \mathcal{M} is known as the *measure theoretic support*. Let us thus write

$$\begin{aligned} d_H(\mathcal{M}) \equiv f(a(1)) &= \lim_{l \rightarrow 0} \frac{1}{\log_2 l} \sum_k p_k(l) \log_2 p_k(l) \\ &\approx \frac{1}{\log_2 \varepsilon} \sum_k p_k(\varepsilon) \log_2 p_k(\varepsilon). \end{aligned} \quad (4.27)$$

Here ε corresponds to a cutoff (or coarse graining) scale of the grid. For the further convenience we will keep $\varepsilon = l_{cut}$ finite throughout all our calculations and set $\varepsilon \rightarrow 0$ only at the end.

In the case of multifractal systems one is often interested in entropy of only certain (uni)fractal subsets. For such a purpose it is useful to introduce a one-parametric family of normalized distributions (zooming or escort distributions) $\varrho(q)$ as

$$\varrho_i(q, l) = \frac{[p_i(l)]^q}{\sum_j [p_j(l)]^q} \sim l^{q a_i - \tau} = l^{f(a_i)}.$$

Because the distribution $\varrho(q, l)$ alters the scaling of the original distribution \mathcal{P}_n , the corresponding measure theoretic support will change. As a matter of fact, distribution $\varrho(q, l)$ enables to form an ensemble of measure theoretic supports $\mathcal{M}^{(q)}$ parametrized by q . Parameter q provides a “zoom in” mechanism to probe various regions of a different singularity exponent. Indeed, from (4.7) we have

$$df(a) = \begin{cases} \leq da & \text{if } q \leq 1 \\ \geq da & \text{if } q \geq 1. \end{cases} \quad (4.28)$$

Integrating (4.28) from $a(q=1)$ to a we obtain

$$f(a) = \begin{cases} \leq a & \text{if } q \leq 1 \\ \geq a & \text{if } q \geq 1, \end{cases} \quad (4.29)$$

and so for $q > 1$ $\varrho(q)$ puts emphasis on the more singular regions of \mathcal{P}_n , while for $q < 1$ the accentuation is on the less singular regions (see also Fig.2). The corresponding

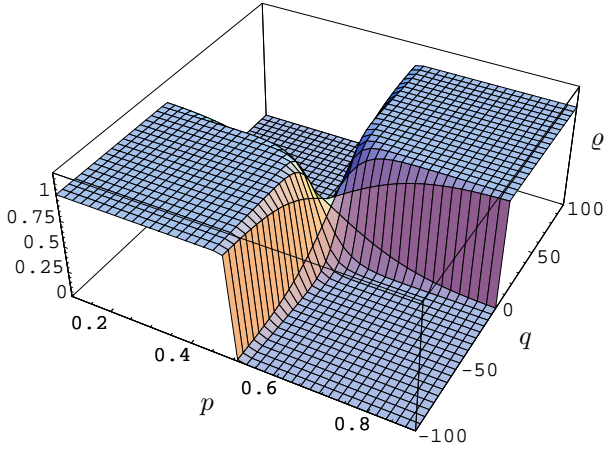


FIG. 2. A plot of the zooming distribution for 2 dimensional \mathcal{P} : $\varrho(q) = p^q / (p^q + (1-p)^q)$.

fractal dimension of the measure theoretic support $\mathcal{M}^{(q)}$ of $\varrho(q)$ is

$$\begin{aligned} d_H(\mathcal{M}^{(q)}) &= \lim_{l \rightarrow 0} \frac{1}{\log_2 l} \sum_k \varrho_k(q, l) \log_2 \varrho_k(q, l) \\ &\approx \frac{1}{\log_2 \varepsilon} \sum_k \varrho_k(q, \varepsilon) \log_2 \varrho_k(q, \varepsilon). \end{aligned} \quad (4.30)$$

We can now use (4.30) to find the promised connection between multifractals and Rényi's entropy. To do this let us observe that the curdling (4.30) mimics the situation occurring in equilibrium statistical physics. There in *canonical formalism* one works with (usually infinite) ensemble of identical macroscopic systems with all possible energy configurations. Notwithstanding only the configurations with $E_i = \langle E \rangle$ dominate in thermodynamic limit.

In fact, defining the “microcanonical” partition function

$$Z_{mic} = \left(\sum_{a_k \in (a_i, a_i + da_i)} 1 \right) = dN(a_i),$$

one gets for $a_i \approx \log_2(p_i) / \log_2 \varepsilon$ (c.f., (4.2))

$$\begin{aligned} \langle a \rangle_{mic} &= \sum_{a_k \in (a_i, a_i + da_i)} \frac{a_k}{Z_{mic}} \approx a_i, \\ \langle f(a) \rangle_{mic} &= \sum_{a_k \in (a_i, a_i + da_i)} \frac{f(a_k)}{Z_{mic}} \approx f(a_i). \end{aligned} \quad (4.31)$$

Because in the micro-canonical approach the distribution is uniform ($\mathcal{E}(a_i) = \{1/dN(a_i)\}$), the corresponding Shannon–Gibbs entropy boils down to the micro-canonical (or Boltzmann) entropy

$$\mathcal{H}(\mathcal{E}(a_i)) = \log_2 dN(a_i) = \log_2 Z_{mic}, \quad (4.32)$$

and hence

$$\frac{\mathcal{H}(\mathcal{E}(a_i))}{\log_2 \varepsilon} \approx -\langle f(a) \rangle_{mic}. \quad (4.33)$$

Interpreting $E_i = -a_i \log_2 \varepsilon$ as “energy” we may define the “inverse temperature” $1/T = \beta / \ln 2$ (note that $k_B = 1 / \ln 2$ here) as

$$1/T = \left. \frac{\partial \mathcal{H}}{\partial E} \right|_{E=E_i} = -\frac{1}{\ln \varepsilon} \frac{\partial Z_{mic}}{\partial a_i} = f'(a_i) = q.$$

Legendre transform then allows to determine the conjugate function $\tau(q)$ via

$$\langle f(a) \rangle_{mic} \approx q \langle a \rangle_{mic} - \tau(q). \quad (4.34)$$

On the other hand, defining the “canonical” partition function as

$$Z_{can} = \sum_i p_i(\varepsilon)^q = \sum_i e^{-\beta E_i},$$

(where the identifications $\beta = q \ln 2$ and $E_i = -\log_2(p_i(\varepsilon))$ are made) the corresponding means are

$$\begin{aligned} a(q) \equiv \langle a \rangle_{can} &= \sum_i \frac{a_i}{Z_{can}} e^{-\beta E_i} \\ &\approx \frac{\sum_i \varrho_i(q, \varepsilon) \log_2 p_i(\varepsilon)}{\log_2 \varepsilon}, \\ f(q) \equiv \langle f(a) \rangle_{can} &= \sum_i \frac{f(a_i)}{Z_{can}} e^{-\beta E_i} \\ &\approx \frac{\sum_i \varrho_i(q, \varepsilon) \log_2 \varrho_i(q, \varepsilon)}{\log_2 \varepsilon}. \end{aligned} \quad (4.35)$$

Let us observe two things. Firstly, the fractal dimension of the measure theoretic support $d_H(\mathcal{M}^{(q)})$ is simply

$f(q)$. If q is a solution of the equation $a_i = \tau'(q)$ then in the “thermodynamic” limit ($\varepsilon \rightarrow 0$) we can identify

$$\begin{aligned} a(q) &= \langle a \rangle_{can} = \langle a \rangle_{mic} \approx a_i, \\ f(q) &= \langle f(a) \rangle_{can} = \langle f(a) \rangle_{mic} \approx f(a_i). \end{aligned} \quad (4.36)$$

Eqs.(4.35) then provide a parametric relationship between $f(q)$ and the singularity exponent $a(q)$. When the parameter q is eliminated one recovers the usual singularity spectrum $f(a)$. Eqs.(4.35) imply that $\langle f \rangle_{can} = q \langle a \rangle_{can} - \tau$, $\langle a \rangle_{can} = d\tau/dq$, and so again the Legendre transform applies. Secondly, because the micro-canonical and canonical entropies coincide in the thermodynamic limit

$$\mathcal{H}(\mathcal{E}(a)) \approx - \sum_k \varrho_k(q, \varepsilon) \log_2 \varrho_k(q, \varepsilon) \equiv \mathcal{H}(\mathcal{P}_n)|_{f(q)}.$$

Here we have used the subscript $f(q)$ to emphasize that the Shannon entropy $\mathcal{H}(\mathcal{P}_n)$ is basically the entropy of an unifractal specified by the fractal dimension $f(q)$ defined in (4.35). Because of relations (4.36) and the Legendre transform (4.7) we obtain after a short algebra

$$\begin{aligned} \frac{\mathcal{H}(\mathcal{P}_n)|_{f(q)}}{\log_2 \varepsilon} + f &= \frac{\mathcal{I}_q}{\log_2 \varepsilon} + \frac{\tau}{q - 1} \\ &- q \left[(\tilde{a} - \langle a \rangle_{can}) + \frac{(\tilde{\tau} - \tau)}{1 - q} \right], \end{aligned} \quad (4.37)$$

with q determined by the condition $\tau'(q) = a$ and

$$\tilde{a} = \frac{\sum_i \varrho_i(q, \varepsilon) \log_2 p_i(\varepsilon)}{\log_2 \varepsilon}, \quad \tilde{\tau} = \frac{\log_2 \sum_i p_i^q(\varepsilon)}{\log_2 \varepsilon}.$$

Applying l’Hospital’s rule we find that

$$\lim_{\varepsilon \rightarrow 0} \left[(\tilde{a} - \langle a \rangle_{can}) + \frac{(\tilde{\tau} - \tau)}{1 - q} \right] \log_2 \varepsilon = 0. \quad (4.38)$$

Multiplying (4.37) by $\log_2 \varepsilon$, taking the small ε limit and employing the renormalization prescriptions (4.10) and (4.14) we finally receive that

$$\mathcal{I}_q^r = \mathcal{H}^r|_{f(q)}. \quad (4.39)$$

The superscript r indicates the renormalized quantities. To understand (4.39) let us note that $\mathcal{H}(\mathcal{P}_n)|_{f(q)}$ can be alternatively written as

$$\begin{aligned} \mathcal{H}(\mathcal{P}_n)|_{f(q)} &\approx \sum_{k=1}^{dN(a)} \frac{p_k(\varepsilon)}{\sum_{l=1}^{dN(a)} p_l(\varepsilon)} \log_2 \left(\frac{p_k(\varepsilon)}{\sum_{l=1}^{dN(a)} p_l(\varepsilon)} \right) \\ &= \log_2 dN(a). \end{aligned} \quad (4.40)$$

Denoting the incomplete distribution $\sum_{k=1}^{dN(a)} p_k(\varepsilon) < 1$ as \mathcal{S} and the conditional distribution $\{p_k(\varepsilon)/\mathcal{S}; k = 1, \dots, dN(a)\}$ as \mathcal{P}'_n then

$$\begin{aligned} \mathcal{H}(\mathcal{P}_n)|_{f(q)} &\approx \mathcal{H}(\mathcal{P}'_n) \\ &= \frac{\sum_{k=1}^{dN(a)} p_k(\varepsilon) \log_2 p_k(\varepsilon)}{\sum_{l=1}^{dN(a)} p_l(\varepsilon)} - \log_2 \frac{1}{\mathcal{S}}. \end{aligned} \quad (4.41)$$

So the RHS of (4.39) equals to Shannon’s information of an incomplete distribution [3,4] minus information corresponding to the total probability of the incomplete system (i.e., unifractal).

In passing we can observe that for $q = 1$ the LHS of (4.39) represents the Shannon entropy of the entire multifractal system, while the RHS stands for the Shannon entropy of the unifractal with the fractal dimension $a(1) = f(a(1)) = D$. It is of course Billingsley’s theorem which makes sure that both sides match in the continuous limit. Now, the passage from multifractals to single-dimensional statistical systems is done by assuming that the a -interval gets infinitesimally narrow and that PDF is smooth. In such a case both a and $f(a)$ collapse to $a = f(a) \equiv D$ and $q = f'(a) = 1$. So, for instance, for a statistical system with a smooth measure and the support space \mathbb{R}^d Eq.(4.39) constitutes a trivial identity. We believe that this is the primary reason why Shannon’s entropy plays such a predominant role in physics of single-dimensional sets.

Let us make finally one more observation. If we apply the MaxEnt approach to a single unifractal (say that with the dimension $f(q)$) and try to infer the most probable incomplete distribution which complies with whatever macroscopic constraints we know about the unifractal subsystem, we have to look for a conditional extremum of Shannon’s entropy $\mathcal{H}(\mathcal{P}_n)|_{f(q)}$. This can be done, at least in principle, in two ways. We can either extremize $\mathcal{H}(\mathcal{P}_n)|_{f(q)}$ with the incomplete distribution keeping \mathcal{S} fixed, or extremize $\mathcal{H}(\mathcal{P}_n)|_{f(q)}$ directly with respect to the zooming distribution $\varrho(q, \varepsilon)$. The second way is often more manageable. As a result we obtain that the least biased incomplete probability distribution on the unifractal characterized by the dimension $f(q)$ is obtained via extremizing Rényi’s entropy $\mathcal{I}_q(\mathcal{P}_n)$ with respect to the zooming distribution $\varrho(q, \varepsilon)$. So by changing the q parameter at Rényi’s entropy one can “skim over” all unifractal Shannon’s entropies. If, additionally, the macroscopic constraints correspond to state variables then MaxEnt approach naturally allows for a thermodynamic description of multifractals.

V. FINAL REMARKS

It was the aim of this paper to present a self-contained discussion of Rényi’s entropy. Apart from formal information theory aspects of Rényi’s entropy we have studied its bearing on various topics of current interest in physics. These include the THC non-extensive entropy,

fractal and multifractal systems, PDF reconstruction theorem, chaotic dynamical systems and MaxEnt approach to thermodynamics.

It should be noted that the thermodynamical or static concept of entropy, though deeply rooted in physics, is rigorously defined only for equilibrium systems or, at best, for adiabatically evolving systems. In fact, the very existence of the entropy in thermodynamics is attributed to Carathéodory's inaccessibility theorem [43] and the statistical interpretation behind the thermodynamical entropy is then usually provided via the ergodic hypothesis [14,44]. When one moves away from equilibrium there are very few clues left of how one should proceed to define entropy. In particular, there is no general concept of ergodicity which could come into our rescue. But just what is entropy then? It is frequently said that entropy is a measure of disorder, and while this needs many qualifications and clarifications it is generally believed that this does represent something essential about it. Insistence on the former interpretation however naturally begs for an operational prescription. To tackle this issue we have resorted to information theory. Here disorder is quantified in terms of missing information and the corresponding information entropy is a measure of our ignorance about a system in question. We feel that the latter is a natural and conceptually very clean extension of the equilibrium concept of entropy. This might be further reinforced by the fact that the information entropy stands a full mathematical rigor. Actually, the information theory provides a whole hierarchy of information entropies each of which is compatible with basic axioms of information theory and theory of probability. Such information entropies are mutually distinguished by their order (Rényi's parameter). It is well known [32] that the information entropy of order 1 (Shannon's entropy) can successfully reproduce the usual equilibrium statistical physics and hence thermodynamics on a simple metric spaces. It was one of the aims of this paper to show that when dealing with (multi)fractal systems one needs to use also information entropies of orders $\alpha \neq 1$ - Rényi entropies. In fact, because the concept of information does not hinge on the notion of equilibrium or non-equilibrium, one may go even further and apply information entropies into various non-equilibrium situations (for $\alpha = 1$ case see e.g., [45] and citations therein).

Because of this versatile nature of Rényi's entropy we are rather tempted to believe that THC entropy is only derived (i.e., not fundamental) concept in physics. We substantiate the latter by arguing that in certain instances - e.g., rare events systems - THC entropy is the leading order approximation to Rényi's entropy. In addition, because Rényi's entropy is a monotonous function of THC entropy all stability conditions in thermodynamics are identical in both cases and so from thermodynamical point of view both entropies are indistinguishable. In those cases it is a matter of taste and/or technical

convenience which one will be applied [6]. It should be also noted that in this light an apparent non-extensivity of THC entropy could be possibly viewed as an artificial (local) feature of much the same origin as is a non-periodicity of leading (i.e., local) contributions to (globally) periodic functions.

It should be, however, admitted that the authors see a possible loophole for THC entropy to play a more pivotal role - i.e., to be an autonomous (not derived) and conceptually clean construct, similarly as, for example, Fisher's entropy⁴ is. The loophole seem to be provided by the *quantum non-locality*. The point is that in order to obtain some breathing space for THC entropy some of the axioms of Rényi's entropy must be bypassed or at least softened. The authors feel that only plausible possibility is to violate the axiom 3 of Section IIA with its additivity of independent information. In fact, we have derived the additivity of entropies for independent experiments with the hidden assumption that experiments are independent if (and only if) they are uncorrelated. In quantum mechanics, however, the relationship between independent and uncorrelated is more delicate. At present it seems that the feasible mechanism which questions, although in a very subtle way, the equivalence between being independent and being uncorrelated is attributed to the quantum non-locality and, in particular the *quantum entanglement*. Bohm-Aharonov effect, Berry phase, EPR paradox, Wheeler's delayed choice experiment or quantum teleportation being the most paramount examples of the aforementioned. Indeed, one can go even so far as to claim that because the whole Universe is inherently quantum correlated one should refrain from using Rényi's entropy altogether. Whether or not these ideas are viable and whether or not the affiliated entropy is connected with THC entropy remains yet to be seen.

As we have shown Rényi's entropy has a build-in predisposition to account for self-similar systems and so it naturally aspires to be an effective tool to describe phase transitions (both in equilibrium and non-equilibrium). It is thus a challenging task to find some connection with such typical tools of critical phenomena physics as are conformal and renormalization groups. The latter could in turn bring about a better understanding of the role of α parameter for systems away from equilibrium. An interesting application of the former observation is in the cosmic string physics. In cosmology, unified gauge theo-

⁴Fisher's entropy (or information) is an important concept in parametric statistics as it represents a measure of the amount of information a given statistical sample contains about the parameter which parametrizes PDF. It is well known that there is an intimate connection between Fisher's and Shannon's [49] (and Rényi's [4]) entropy, yet both concepts are completely autonomous.

ries of particle interactions allow for a sequence of phase transitions in the very early universe some of which may lead to defect formation via the so called Kibble–Zurek mechanism [50]. Cosmic strings as the most pronounced example of such defects, could have important relevance on the large scale structure formation of the universe or on cosmic microwave background radiation anisotropies. In astrophysics, for instance, cosmic strings could play an important rôle in dynamics of neutron stars and in the galaxy astrophysics. In usual cases when the grand-canonical approach is applied it is argued that at the critical (phase transition) temperature at which strings tend to fragment into smallest allowed loops, while large loops become exponentially suppressed - i.e., at Hagedorn temperature [51], the correspondence between the canonical and micro-canonical ensembles breaks down as the grand-canonical partition function diverges [52]. Various viewpoints with different remedies were lately proposed in the literature. It seems, however, that non of the treatments has accommodated the well known fact that the string state-space acquires approximately self-similar structure which is exact at critical temperature [51,52]. From this standpoint Rényi's statistics appears to be particularly suitable for generalization of the Hagedorn theory as it could better grasp the vital features near the critical point. In addition, Rényi's theory can be applied to construct the generalized grand-canonical partition function for the string network. Our current results suggest that the new phase transition temperature should be lower than the one predicted by Hagedorn's theory. It would be definitely interesting to exploit this further and contrast our way with the more customary conformal theory approach. Work along those lines is presently in progress [53].

Let us finally mention that because symmetry breaking phase transitions with string-like defects occur in a variety of physical systems ranging from ^3He and ^4He superfluids to the early Universe, with superconductors and liquid crystals in between, one can hope that predictions based on Rényi's entropy could be directly tested in laboratory. In this connection, the analysis of vortex tangle [54] (turbulence of vortex loops in superfluid phase of ^4He) is one such particularly promising systems with the room-size experimental setting, (see e.g., [55]).

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APPENDIX A

In this appendix we present an alternative way of finding the unique class of the Kolmogorov–Nagumo functions. Let us start with Eq.(2.10) which we rewrite in the form

$$f(\zeta x) = a(x)f((\zeta - 1)x) + f(x), \quad (5.1)$$

with ζ being an arbitrary real constant ($\zeta \geq 0$). The latter is equivalent to the equation

$$f(\zeta x) = \frac{1 - a^\zeta(x)}{1 - a(x)} f(x). \quad (5.2)$$

Note that when $\zeta \rightarrow 0$ then $f(0) = 0$. The latter should be imposed as a boundary condition on prospective solutions. The solution of the functional equation (5.2) can be easily found, indeed realizing that functions fulfilling the scaling condition (5.2) obey the Euler-type equation

$$x \frac{\partial}{\partial x} f(x) = - \frac{a(x) \ln a(x)}{1 - a(x)} f(x), \quad (5.3)$$

we may directly write that

$$f(x) = \gamma \exp \left(- \int dx \frac{a(x) \ln a(x)}{x(1 - a(x))} \right). \quad (5.4)$$

Shortly we will see that function (5.4) is the only one fulfilling the functional equation (5.1). Let us, however, first determine the function $a(x)$. From (5.1) follows that

$$a(x) = \frac{f(\zeta x) - f(x)}{f((\zeta - 1)x)}. \quad (5.5)$$

Because the latter should be true for any $\zeta \geq 0$ we may safely assume that $\zeta = 1 + \varepsilon/x$ with ε being an infinitesimal. Then with a help of l'Hospital rule we obtain

$$a(x) = \frac{f'(x)}{f'(0)}, \Rightarrow f(x) = f'(0) \int_0^x dy a(y). \quad (5.6)$$

Note that $a(0) = 1$. On the other hand (5.5) may be equivalently written as

$$a((\zeta - 1)x) = \frac{f(\zeta x) - f((\zeta - 1)x)}{f(x)}. \quad (5.7)$$

Taking now derivative $\partial/\partial(\zeta - 1)$, using (5.1) and setting successively $\zeta = 2$ we get

$$\begin{aligned} a'(x) &= (a(x) - 1) (\ln f(x))' = \frac{a(x) \ln a(x)}{x}, \\ \Rightarrow \ln a(x) &= cx. \end{aligned} \quad (5.8)$$

If the integration constant $c \neq 0$ then $a(x) = \exp(cx)$ and hence (see (5.4) and (5.6))

$$f(x) = \gamma (\exp(cx) - 1). \quad (5.9)$$

In the latter the condition $f(0) = 0$ was used. We have defined that $\gamma = f'(0)/c$. In case that $c = 0$, we have from (5.8) that $a(x) = \text{const.} = 1$ and so

$$f(x) = f'(0)x. \quad (5.10)$$

So we see that the compatible Kolmogorov–Nagumo functions are only linear and exponential ones. We should also note that the linear $f(x)$ is retrieved from the exponential $f(x)$ in the limit $c \rightarrow 0$.

Let us now turn to the point of uniqueness of $f(x)$. For that purpose let us assume that there are two different functions $f_1(x)$ and $f_2(x)$ both fulfilling the equation (5.1) with an identical $a(x)$ and arbitrary $\zeta \geq 0$, i.e.,

$$\begin{aligned} f_1(\zeta x) &= a(x)f_1((\zeta - 1)x) + f_1(x), \\ f_2(\zeta x) &= a(x)f_2((\zeta - 1)x) + f_2(x). \end{aligned} \quad (5.11)$$

Because the latter should hold for any $\zeta \geq 0$ the following must be true

$$\begin{aligned} a'(x) &= (a(x) - 1)(\ln f_1(x))' \\ &= (a(x) - 1)(\ln f_2(x))'. \end{aligned} \quad (5.12)$$

As a result we have that $(\ln f_1(x))' = (\ln f_2(x))'$ and so $f_1(x) = \text{const.} \times f_2(x)$, which confirms that only linear and exponential functions are compatible with the additivity of information.

APPENDIX B

Here we present a proof that the five postulates of Section IIA determine uniquely both Shannon's and Rényi's entropies. Our proof consists of four steps:

a) Let us denote first $\mathcal{I}(1/n, \dots, 1/n) = \mathcal{L}(n)$. Then from the second and fifth axiom follows that

$$\begin{aligned} \mathcal{L}(n) &= \mathcal{I}(1/n, \dots, 1/n, 0) \\ &\leq \mathcal{I}(1/(n+1), \dots, 1/(n+1)) = \mathcal{L}(n+1), \end{aligned} \quad (5.13)$$

i.e., \mathcal{L} is a non-decreasing function.

b) To find the explicit form of \mathcal{L} we employ the third postulate. For this purpose we will assume that we have m mutually independent experiments $\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(m)}$ each with r equally probable outcomes, so

$$\mathcal{I}(\mathcal{A}^{(k)}) = \mathcal{I}(1/r, \dots, 1/r) = \mathcal{L}(r), \quad (1 \leq k \leq m). \quad (5.14)$$

Because experiments are independent $\mathcal{I}(\mathcal{A}^{(k)}|\mathcal{A}^{(l)}) = \mathcal{A}_i^{(l)} = \mathcal{I}(\mathcal{A}^{(k)})$ for $k \neq l$ and $\forall i$, axiom 3 (generalized to the case of m experiments) implies that

$$\begin{aligned} \mathcal{I}(\mathcal{A}^{(1)} \cap \mathcal{A}^{(2)} \cap \dots \cap \mathcal{A}^{(m)}) &= \sum_{k=1}^m \mathcal{I}(\mathcal{A}^{(k)}) \\ &= m\mathcal{L}(r). \end{aligned} \quad (5.15)$$

On the other hand, the experiment $\mathcal{A}^{(1)} \cap \mathcal{A}^{(2)} \cap \dots \cap \mathcal{A}^{(m)}$ consists of r^m equally probable outcomes, and so

$$\mathcal{L}(r^m) = m\mathcal{L}(r). \quad (5.16)$$

This is nothing but Cauchy's functional equation [13]. It might be shown [13,22] that for non-decreasing functions (5.16) has a unique solution; $\mathcal{L}(r) = \kappa \ln(r)$. The constant κ can be determined from the axiom 2 which then directly implies that $\mathcal{L}(r) = \log_2(r)$.

c) We now determine $\mathcal{I}(\mathcal{P})$ using the axiom 3. To this extent we will assume that the experiment $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n)$ is described by the distribution $\mathcal{P} = \{p_1, p_2, \dots, p_n\}$ with p_k ($1 \leq k \leq n$) being rational numbers, say

$$p_k = \frac{g_k}{g}, \quad \sum_{k=1}^n g_k = g, \quad g_k \in \mathbb{N}. \quad (5.17)$$

Let us have further an experiment $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_g)$ and let $\mathcal{Q} = \{q_1, q_2, \dots, q_g\}$ is the associated distribution. We split $(\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_g)$ into n groups containing g_1, g_2, \dots, g_n events respectively. Consider now a particular situation in which whenever event \mathcal{A}_i in \mathcal{A} happens then in \mathcal{B} all the g_k events of k -th group occur with the equal probability $1/g_k$ and all the other events in \mathcal{B} have probability zero. Hence

$$\mathcal{I}(\mathcal{B}|\mathcal{A} = \mathcal{A}_k) = \mathcal{I}(1/g_k, \dots, 1/g_k) = \log_2 g_k, \quad (5.18)$$

and so

$$\mathcal{I}(\mathcal{B}|\mathcal{A}) = f^{-1} \left(\sum_{k=1}^n \varrho_k(\alpha) f(\log_2 g_k) \right). \quad (5.19)$$

On the other hand, $\mathcal{I}(\mathcal{A} \cap \mathcal{B})$ can be directly evaluated. Realizing that the joint probability distribution corresponding to $\mathcal{A} \cap \mathcal{B}$ is

$$\begin{aligned} \mathcal{R} &= \{r_{kl} = p_k q_{l|k}\} \\ &= \left\{ \underbrace{\frac{p_1}{g_1}, \dots, \frac{p_1}{g_1}}_{g_1 \times}, \underbrace{\frac{p_2}{g_2}, \dots, \frac{p_2}{g_2}}_{g_2 \times}, \dots, \underbrace{\frac{p_n}{g_n}, \dots, \frac{p_n}{g_n}}_{g_n \times} \right\} \\ &= \{1/g, \dots, 1/g\}, \end{aligned} \quad (5.20)$$

we obtain that $\mathcal{I}(\mathcal{A} \cap \mathcal{B}) = \mathcal{L}(g) = \log_2 g$. Applying the axiom 3 then

$$\begin{aligned} \mathcal{I}(\mathcal{P}) &= \log_2 g - f^{-1} \left(\sum_{k=1}^n \varrho_k(\alpha) f(\log_2 g_k) \right) \\ &= \log_2 g - f^{-1} \left(\sum_{k=1}^n \varrho_k(\alpha) f(\log_2 p_k + \log_2 g) \right) \\ &= \mathcal{L}(g) - f^{-1} \left(\sum_{k=1}^n \varrho_k(\alpha) f(\log_2 p_k + \mathcal{L}(g)) \right). \end{aligned} \quad (5.21)$$

Let us define $f_y(x) = f(-x - y) (\Rightarrow f^{-1}(x) + y = -f_y^{-1}(x))$. Then

$$\mathcal{I}(\mathcal{P}) = f_{\mathcal{L}(g)}^{-1} \left(\sum_{k=1}^n \varrho_k(\alpha) f_{\mathcal{L}(g)}(\mathcal{I}_k) \right). \quad (5.22)$$

By axiom 4 $f(x)$ is invertible in $[0, \infty)$ and so both $f_{\mathcal{L}(g)}$ and $f_{\mathcal{L}(g)}^{-1}$ are continuous on $[0, \infty)$. Applying now the postulate 1 (axiom of continuity) we may extend the result (5.22) from rational p_k 's to any real valued p_k 's defined in $[0, 1]$.

Let us consider now the case of independent events (i.e., $\mathcal{I}(\mathcal{B}|\mathcal{A}) = \mathcal{I}(\mathcal{B})$). From Section IIA (and/or Appendix A) we already know that in this case the only candidate for $f_{\mathcal{L}(g)}$ is a linear function or a linear function of an exponential function. Bearing in mind that two functions which are linear functions of each other give the same mean (see Section IIA) we may choose either $f_{\mathcal{L}(g)}(x) = x$ or $f_{\mathcal{L}(g)}(x) = 2^{(\lambda-1)x}$, $\lambda \neq 1$. Consequently from (5.22) we may write

$$\begin{aligned} \mathcal{I}(\mathcal{P}) &= \frac{1}{(\lambda-1)} \log_2 \left(\sum_{k=1}^n p_k^{\alpha-\lambda+1} \right) \\ &+ \frac{1}{(1-\lambda)} \log_2 \left(\sum_{k=1}^n p_k^\alpha \right). \end{aligned} \quad (5.23)$$

It should be also noticed that from the axiom 5 follows that $(\alpha - \lambda + 1) > 0$ and $\alpha > 0$. Within the scope of previous inequalities Eq.(5.23) is valid for any λ . It should be particularly noticed that $\mathcal{I}(\mathcal{P})$ is continuous at $\lambda = 1$ as both the left and right limit coincide. It can be easily checked that $\lambda = 1$ corresponds precisely to the case of $f_{\mathcal{L}(g)}(x) = x$. Quantity (5.23) was firstly proposed by Kapur [56] and named the *entropy of order $2 - \lambda$ and type α* .

Finally, it should be born in mind that because the mean (5.19) is unchanged under linear transformation of function $f(x)$ we could, from the very beginning, restrict ourselves to only *positive* invertible functions on $[0, \infty)$.

d) In the last step we will specify the relationship between α and λ . Using the fact that the experiment $\mathcal{A} \cap \mathcal{B}$ has the (joint) probability distribution $\mathcal{R} = \{r_{kl} = p_k q_{l|k}\}$ we have

$$\begin{aligned} \mathcal{I}(\mathcal{A} \cap \mathcal{B}) &= \frac{1}{(\lambda-1)} \log_2 \left(\sum_{k,l} (p_k q_{l|k})^{\alpha-\lambda+1} \right) \\ &+ \frac{1}{(1-\lambda)} \log_2 \left(\sum_{k,l} (p_k q_{l|k})^\alpha \right), \end{aligned} \quad (5.24)$$

and

$$\begin{aligned} \mathcal{I}(\mathcal{B}|\mathcal{A}) &= \frac{1}{(\lambda-1)} \log_2 \left(\sum_k p_k^\alpha \right) \\ &+ \frac{1}{(1-\lambda)} \log_2 \left(\sum_k p_k^\alpha \frac{\sum_l (q_{l|k})^\alpha}{\sum_l (q_{l|k})^{\alpha-\lambda+1}} \right). \end{aligned} \quad (5.25)$$

Eq.(5.25) is a result of the fact that

$$2^{(1-\lambda)\mathcal{I}(\mathcal{B}|\mathcal{A}=\mathcal{A}_k)} = \frac{\sum_l (q_{l|k})^\alpha}{\sum_l (q_{l|k})^{\alpha-\lambda+1}},$$

and that $f_{\mathcal{L}(g)}(x) = 2^{(\lambda-1)x} \Rightarrow f(x) = 2^{(1-\lambda)x}$. Combining the axiom 3 and Eqs.(5.24)–(5.25) we obtain for $\lambda \neq 1$ the identity

$$\begin{aligned} &\frac{\sum_k p_k^{\alpha-\lambda+1} \sum_l (q_{l|k})^{\alpha-\lambda+1}}{\sum_k p_k^\alpha \sum_l (q_{l|k})^\alpha} \\ &= \frac{\sum_k p_k^{\alpha-\lambda+1}}{\sum_k p_k^\alpha \frac{\sum_l (q_{l|k})^\alpha}{\sum_l (q_{l|k})^{\alpha-\lambda+1}}}. \end{aligned} \quad (5.26)$$

Introducing the random variable

$$\mathcal{Q}^{(\alpha\lambda)} = \left\{ \sum_l (q_{l|k})^{\alpha-\lambda+1} \right\},$$

we may equivalently rewrite (5.26) as

$$\begin{aligned} &\frac{\sum_{k,l} r_{kl}^{\alpha-\lambda+1}}{\sum_{k,l} r_{kl}^\alpha} = \frac{\sum_{k,l} r_{kl}^{\alpha-\lambda+1} / \mathcal{Q}_k^{(\alpha\lambda)}}{\sum_{k,l} r_{kl}^\alpha / \mathcal{Q}_k^{(\alpha\lambda)}}, \\ &\Leftrightarrow \langle 1/\mathcal{Q}^{(\alpha\lambda)} \rangle_\alpha = \langle 1/\mathcal{Q}^{(\alpha\lambda)} \rangle_{\alpha-\lambda+1}. \end{aligned} \quad (5.27)$$

Here $\langle \dots \rangle_x$ is defined with respect to the distribution

$$\mathcal{P} = \left\{ \sum_l (r_{kl})^x / \sum_{k,l} (r_{kl})^x \right\}.$$

Because p_k 's are arbitrary, equality (5.27) happens if and only if $\mathcal{Q}^{(\alpha\lambda)}$ is a constant [19]. The latter implies that

$$\sum_l (q_{l|k})^{\alpha-\lambda+1} = \text{const.}, \quad \text{for } \forall k \text{ and } \forall q_{l|k}. \quad (5.28)$$

It is easy to see that Eq.(5.28) is satisfied only when $\alpha = \lambda$. Substituting $\lambda = \alpha$ into (5.23) we find

$$\mathcal{I}(\mathcal{P}) = \mathcal{I}(\mathcal{A}) = \frac{1}{1-\alpha} \log_2 \sum_k (p_k)^\alpha. \quad (5.29)$$

The proof for $\lambda = 1$ follows the analogous route. This proves our assertion.

APPENDIX C

In this appendix we derive some basic properties of the information measure $\mathcal{I}_\alpha(\mathcal{B}|\mathcal{A})$.

From Appendix B we know that $f(x)$ compatible with axioms 1–5 is (up to a linear combination) either x or $2^{(1-\alpha)x}$. Then $\mathcal{I}(\mathcal{B}|\mathcal{A})$ appearing in the axiom 3 turns out to have the form

$$\mathcal{I}_\alpha(\mathcal{B}|\mathcal{A}) = \frac{1}{(1-\alpha)} \log_2 \left(\frac{\sum_{k,l} (r_{kl})^\alpha}{\sum_k p_k^\alpha} \right), \quad (5.30)$$

with $\mathcal{P}(\mathcal{A} \cap \mathcal{B}) = \{r_{kl} = p_k q_{l|k} = q_l p_{k|l}\}$. We have reintroduced the sub-index α to emphasize the parametric dependence of \mathcal{I} . It results from (5.30) that for every α

$$0 \leq \mathcal{I}_\alpha(\mathcal{B}|\mathcal{A}) \leq \log_2 n, \quad (5.31)$$

where n is the number of outcomes in the experiment \mathcal{B} . Indeed, $0 \leq \mathcal{I}_\alpha(\mathcal{B}|\mathcal{A})$ holds due to a simple fact that for a fixed k and $\alpha > 1$

$$\sum_l (r_{kl})^\alpha = p_k^\alpha \sum_l (q_{l|k})^\alpha \leq p_k^\alpha, \quad (5.32)$$

(realize that $\sum_l q_{l|k} = 1$). Equality in (5.32) is clearly valid if and only if for any k there exists just one $l = l(k)$ such that $q_{l(k)|k} = 1$ and 0 otherwise. The latter means that outcomes of \mathcal{A} uniquely determine outcomes of \mathcal{B} and hence we do not learn any new information about \mathcal{B} by knowing \mathcal{A} . In such a case (5.30) gives $\mathcal{I}_\alpha(\mathcal{B}|\mathcal{A}) = 0$. This is what one would naturally expect from a conditional information.

Similarly, for $0 < \alpha < 1$ the reverse inequality in (5.32) holds and hence $\sum_l (r_{kl})^\alpha \geq p_k^\alpha$ (former comments about the equality apply here as well). This proves our assertion about the LHS inequality in (5.31).

On the other hand, the RHS inequality in Eq.(5.31) holds because for $\alpha > 1$, $\sum_l (q_{l|k})^\alpha$ is a convex function which has its minimum at $q_{l|k} = 1/n$ (for $\forall l, k$). So

$$\sum_l (q_{l|k})^\alpha \geq n^{1-\alpha},$$

while for $0 < \alpha < 1$ the opposite inequality holds. Thus

$$\begin{aligned} \mathcal{I}_\alpha(\mathcal{B}|\mathcal{A}) &= \frac{1}{(1-\alpha)} \log_2 \left(\frac{\sum_k p_k^\alpha \sum_l (q_{l|k})^\alpha}{\sum_k p_k^\alpha} \right) \\ &\leq \log_2 n. \end{aligned} \quad (5.33)$$

Inequality (5.33) may be viewed as a weak version of the well known $\alpha = 1$ case where $\mathcal{H}(\mathcal{B}|\mathcal{A}) \leq \mathcal{H}(\mathcal{B})$ with equality if and only if \mathcal{B} and \mathcal{A} are independent experiments [22] (i.e., knowing outcomes of \mathcal{A} does not have any effect on the distribution of outcomes of \mathcal{B}). However, aforesaid does not generally hold for $\alpha \neq 1$. This is because

$$\mathcal{I}_\alpha(\mathcal{B}) - \mathcal{I}_\alpha(\mathcal{B}|\mathcal{A}) = \frac{1}{(1-\alpha)} \log_2 \left(\frac{\sum_{l,k} p_k^\alpha q_l^\alpha}{\sum_{l,k} (p_k q_{l|k})^\alpha} \right), \quad (5.34)$$

and the identity

$$\left(\sum_{k,l} (p_k q_l)^\alpha \right)^{1/(1-\alpha)} = \left(\sum_{k,l} (p_k q_{l|k})^\alpha \right)^{1/(1-\alpha)}, \quad (5.35)$$

can be fulfilled for $\alpha \neq 1$ in numerous ways [26] without assuming that $q_{l|k} = q_l$ (for example, in the $\alpha = 2$ case we may chose; $\mathcal{P} = \{1/n\}$, $\mathcal{Q} = \{1/n\}$ and $\mathcal{P}(\mathcal{B}|\mathcal{A}) = \{1, 0, 0, \dots, 0\}$). However, in the limiting case $\alpha \rightarrow 1$ Eq.(5.35) turn out to be

$$2^{-\sum_{k,l} p_k q_l \log_2(p_k q_l)} = 2^{-\sum_{k,l} r_{kl} \log_2(r_{kl})}, \quad (5.36)$$

which has the solution if and only if $q_{l|k} = q_l$, i.e., in the case of independent events [22]. Yet still, $\mathcal{I}_\alpha(\mathcal{B}|\mathcal{A})$, $\alpha \neq 1$ can be, in a sense, viewed as conditional information. This is so because when \mathcal{B} and \mathcal{A} are independent then from (5.34) follows that $\mathcal{I}_\alpha(\mathcal{B}) = \mathcal{I}_\alpha(\mathcal{B}|\mathcal{A})$. Opposite implication, as we have seen, is not valid in general. The opposite implication is, however, valid when \mathcal{B} has an equiprobable distribution. The latter is a simple consequence of Jensen's inequality because for $\alpha > 1$

$$p_k^\alpha = \left(\sum_l q_l \frac{r_{kl}}{q_l} \right)^\alpha \leq \sum_l q_l \left(\frac{r_{kl}}{q_l} \right)^\alpha = \sum_l q_l (p_{k|l})^\alpha,$$

and so for $\mathcal{P}(\mathcal{B}) = \mathcal{Q} = \{q_l = 1/n\}$

$$\begin{aligned} \frac{\sum_{l,k} q_l^\alpha p_k^\alpha}{\sum_{k,l} (r_{kl})^\alpha} &\leq \frac{\sum_l q_l^\alpha \sum_{j,k} q_j (p_{k|j})^\alpha}{\sum_{l,k} q_l^\alpha (p_{k|l})^\alpha} = 1, \\ \Rightarrow \mathcal{I}_\alpha(\mathcal{B}) - \mathcal{I}_\alpha(\mathcal{B}|\mathcal{A}) &\geq 0, \end{aligned} \quad (5.37)$$

with equality if and only if the equality in Jensen's inequality holds. This happens only when $p_{k|l}$ is a constant for $\forall l$, i.e., when \mathcal{A} and \mathcal{B} are independent. Counterpart with $0 < \alpha < 1$ can be proved in exactly the same way.

APPENDIX D

In this appendix we derive relations (3.4) and (3.5). We begin with the notion of the integration of continuous functions defined on fractal sets [47,48]. Consider a fractal set M embedded in a d -dimensional space. Let us cover the set with a mesh $M^{(l)}$ of d -dimensional (disjoint) cubes $M_i^{(l)}$ of size l^d and let $N_l(M)$ is a minimal number of the cubes needed for the covering. Functions with the support in the mesh are called *simple* if they can be decomposed in the following way:

$$\mathcal{G}^{(l)}(\mathbf{x}) = \sum_i^{N_l} \mathcal{G}_i^{(l)} \chi_i^{(l)}(\mathbf{x}). \quad (5.38)$$

Here $\chi_i^{(l)}$ are characteristic functions, i.e.

$$\chi_i^{(l)} = \begin{cases} 1 & \text{if } \mathbf{x} \in M_i^{(l)} \\ 0 & \text{if } \mathbf{x} \notin M_i^{(l)} \end{cases}. \quad (5.39)$$

Then the integral $\int_M d\mu \mathcal{G}^{(l)}$ is defined as

$$\int_M d\mu(\mathbf{x}) \mathcal{G}^{(l)}(\mathbf{x}) = \sum_i^{N_l} \mathcal{G}_i^{(l)} \mu^{(l)}(M_i^{(l)}), \quad (5.40)$$

where the measure $\mu^{(l)}$ is the measure on the covering mesh. The precise form of the measure will be specified shortly. On the covering mesh $M^{(l)}$ we can build a σ -structure in a usual way. As a result, if \mathcal{G} is a nonnegative $\mu^{(l)}$ measurable function then $\mathcal{G}(\mathbf{x}) = \lim_{l \rightarrow 0} \mathcal{G}^{(l)}(\mathbf{x})$ for all $\mathbf{x} \in M^{(l)}$, for some sequence $\{\mathcal{G}_i^{(l)}\}$ of monotonic increasing nonnegative simple functions. Owing to this fact we may define

$$\int_M d\mu(\mathbf{x}) \mathcal{G}(\mathbf{x}) = \lim_{l \rightarrow 0} \sum_{i=1}^{N_l} \mathcal{G}_i^{(l)} \mu^{(l)}(M_i^{(l)}). \quad (5.41)$$

In this connection it is important to notice that due to the scaling prescription (4.1)

$$\log l^D = -\log N_l + o(l^0) \Rightarrow l^D N_l = V_l \rightarrow V. \quad (5.42)$$

Here V_l is the pre-fractal volume which in the small l limit converges to the true fractal volume V . Natural candidate for $\mu^{(l)}(M_i^{(l)})$ is the fraction $V(M_i^{(l)})/N_l$ which in the small l limit behaves as⁵: $l^D = n^{-D}$. So particularly when \mathcal{F} is a continuous PDF we have

$$\int_M d\mu(\mathbf{x}) \mathcal{F}(\mathbf{x}) = \lim_{l \rightarrow 0} \sum_{i=1}^{N_l} \mathcal{F}_i^{(l)} l^D. \quad (5.43)$$

The integrated probability of the k -th cube is thus $p_{nk} = \mathcal{F}_k^{(l)} l^D$. A simple consistency check can be demonstrated on $p_{nk} = \mathcal{E}_{nk}$. Indeed, from Section IV A we know that $\mathcal{E}_{nk} = l^D/V_l$ and so may write

$$1 = \lim_{l \rightarrow 0} \sum_{k=1}^{N_l} \mathcal{E}_{nk} = \lim_{l \rightarrow 0} \sum_{k=1}^{N_l} \frac{l^D}{V_l} = \int_M d\mu \frac{1}{V} = 1. \quad (5.44)$$

We thus see that the integral prescription (5.43) applies correctly in the case of uniform distributions.

Using now the renormalization prescription (3.4)

$$\begin{aligned} \tilde{\mathcal{I}}_\alpha(\mathcal{F}) &\equiv \lim_{l \rightarrow 0} (\mathcal{I}_\alpha(\mathcal{P}_n) - \mathcal{I}_\alpha(\mathcal{E}_n)) \\ &= \lim_{l \rightarrow 0} \left(\frac{1}{1-\alpha} \log_2 \left(\sum_{i=1}^{N_l} (\mathcal{F}_i^{(l)} l^D)^\alpha / \sum_{i=1}^{N_l} \frac{l^{D\alpha}}{V_l^\alpha} \right) \right) \\ &= \lim_{l \rightarrow 0} \left(\frac{1}{1-\alpha} \log_2 \left(\sum_{i=1}^{N_l} (\mathcal{F}_i^{(l)})^\alpha l^D / \sum_{i=1}^{N_l} \frac{l^D}{V_l^\alpha} \right) \right) \\ &= \frac{1}{1-\alpha} \log_2 \left(\frac{\int_M d\mu \mathcal{F}^\alpha}{\int_M d\mu 1/V^\alpha} \right). \end{aligned} \quad (5.45)$$

If we use the renormalization prescription (3.5) (or equivalently when we set $V = 1$ for $\mathcal{I}_\alpha(\mathcal{E}_n)$ in (5.45)) we easily see that

$$\begin{aligned} \mathcal{I}_\alpha(\mathcal{F}) &\equiv \lim_{l \rightarrow 0} (\mathcal{I}_\alpha(\mathcal{P}_n) - \mathcal{I}_\alpha(\mathcal{E}_n)|_{V=1}) \\ &= \lim_{l \rightarrow 0} (\mathcal{I}_\alpha(\mathcal{P}_n) + D \log_2 l) \\ &= \frac{1}{1-\alpha} \log_2 \left(\int_M d\mu \mathcal{F}^\alpha \right). \end{aligned} \quad (5.46)$$

Our renormalization prescription is obviously consistent only when integrals on the RHS of (5.45) and (5.46) exist.

APPENDIX E

We show here that Rényi's entropy $\mathcal{I}_\alpha^{(d)}(\mathcal{F})$ is not invariant under a transformation of the continuous random variable $\mathcal{A}^{(d)}$ while $\tilde{\mathcal{I}}_\alpha^{(d)}(\mathcal{F})$ is. Note first that in a discrete case, outcomes $\mathcal{A}_1, \dots, \mathcal{A}_n$ have the same probability distribution p_1, \dots, p_n as outcomes $h(\mathcal{A}_1), \dots, h(\mathcal{A}_n)$, where $h(\dots)$ is an arbitrary "well behaved" function. Hence Rényi's entropy for such a system is invariant under the h -transformation. However, in the continuous case even the simplest linear transformation $\mathcal{A}^{(d)} \rightarrow c\mathcal{A}^{(d)}$ does not leave $\mathcal{I}_\alpha^{(d)}(\mathcal{F})$ invariant, indeed after rescaling $\mathcal{A}^{(d)}$ to $c\mathcal{A}^{(d)}$ we obtain

$$\begin{aligned} (c\mathcal{A}^{(d)})_n &\equiv \tilde{\mathcal{A}}_n^{(d)} \\ &= \left(c \frac{[(nc)\mathcal{A}_1]}{(nc)}, c \frac{[(nc)\mathcal{A}_2]}{(nc)}, \dots, c \frac{[(nc)\mathcal{A}_d]}{(nc)} \right) \\ &= c\mathcal{A}_{(nc)}^{(d)}, \end{aligned}$$

and so

$$\begin{aligned} \mathcal{I}_\alpha^{(d)}(c\mathcal{A}^{(d)}) &= \lim_{n \rightarrow \infty} \left(\mathcal{I}_\alpha(\tilde{\mathcal{A}}_n^{(d)}) - d \log_2 n \right) \\ &= \lim_{(nc) \rightarrow \infty} \left(\mathcal{I}_\alpha(c\mathcal{A}_{(nc)}^{(d)}) - d \log_2(nc) + d \log_2 c \right) \\ &= \mathcal{I}_\alpha^{(d)}(\mathcal{A}^{(d)}) + d \log_2 c. \end{aligned} \quad (5.47)$$

So $\mathcal{I}_\alpha^{(d)}(c\mathcal{A}^{(d)}) \neq \mathcal{I}_\alpha^{(d)}(\mathcal{A}^{(d)})$. Situation becomes, however, different when we consider $\tilde{\mathcal{I}}_\alpha^{(d)}(c\mathcal{A}^{(d)})$. This is because we can rewrite $\tilde{\mathcal{I}}_\alpha^{(d)}(c\mathcal{A}^{(d)})$ as

$$\begin{aligned} \tilde{\mathcal{I}}_\alpha^{(d)}(c\mathcal{A}^{(d)}) &= \lim_{n \rightarrow \infty} \left(\mathcal{I}_\alpha(\tilde{\mathcal{A}}_n^{(d)}) - d \log_2 n \right) \\ &\quad - \lim_{n \rightarrow \infty} \left(\mathcal{I}_\alpha(\mathcal{E}_{(nc)}^{(d)}) - d \log_2 n \right). \end{aligned} \quad (5.48)$$

Here we have used $\mathcal{E}_{(nc)}^{(d)}$ instead of $\mathcal{E}_n^{(d)}$ because the rescaling changes also the volume V of the outcome space into cV . A simple consequence of Eq.(5.48) is that $\tilde{\mathcal{I}}_\alpha^{(d)}(c\mathcal{A}^{(d)}) = \tilde{\mathcal{I}}_\alpha^{(d)}(\mathcal{A}^{(d)})$. In fact, when $h = (h_1, \dots, h_d)$ is an invertible and differentiable (vector) function it is simple to rewrite $\tilde{\mathcal{I}}_\alpha^{(d)}(\mathcal{F})$ in a fully covariant manner. Indeed, realizing that scalar density transforms as

$$\mathcal{F}(\mathbf{x}) = \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right| \hat{\mathcal{F}}(\mathbf{y}). \quad (5.49)$$

⁵It should be noted that the measure just defined basically coincides with the D -dimensional Hausdorff measure.

(here $\mathbf{y} = h(\mathbf{x})$) we also know that

$$1/V = \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right| m(\mathbf{y}), \quad (5.50)$$

(here $m(\mathbf{y})$ denotes the h -transformed uniform PDF). Then we see that

$$\begin{aligned} \tilde{\mathcal{I}}_\alpha^{(d)}(\mathcal{A}^{(d)}) &= \frac{1}{(1-\alpha)} \log_2 \left(\int_V d^d \mathbf{x} \mathcal{F}^\alpha(\mathbf{x}) V^{\alpha-1} \right) \\ &= \frac{1}{(1-\alpha)} \log_2 \left(\int_{h(V)} d^d \mathbf{y} \left(\frac{\hat{\mathcal{F}}(\mathbf{y})}{m(\mathbf{y})} \right)^\alpha m(\mathbf{y}) \right) \\ &= \tilde{\mathcal{I}}_\alpha^{(d)}(h(\mathcal{A}^{(d)})). \end{aligned} \quad (5.51)$$

If h_1 and h_2 are any two invertible and differentiable vector functions so is their composition $h_2 \circ h_1$ and then

$$\begin{aligned} \tilde{\mathcal{I}}_\alpha^{(d)}(\mathcal{A}^{(d)}) &= \tilde{\mathcal{I}}_\alpha^{(d)}(h_1(\mathcal{A}^{(d)})) \\ &= \frac{1}{(1-\alpha)} \log_2 \left(\int_{h_1(V)} d^d \mathbf{y} \left(\frac{\mathcal{F}_1(\mathbf{y})}{m_1(\mathbf{y})} \right)^\alpha m_1(\mathbf{y}) \right) \\ &= \frac{1}{(1-\alpha)} \log_2 \left(\int_{h_2 \circ h_1(V)} d^d \mathbf{z} \left(\frac{\mathcal{F}_2(\mathbf{z})}{m_2(\mathbf{z})} \right)^\alpha m_2(\mathbf{z}) \right) \\ &= \tilde{\mathcal{I}}_\alpha^{(d)}(h_2 \circ h_1(\mathcal{A}^{(d)})), \end{aligned} \quad (5.52)$$

with

$$\begin{aligned} \mathcal{F}_1(\mathbf{y}) \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right| &= \mathcal{F}(\mathbf{x}), \quad \mathcal{F}_2(\mathbf{z}) \left| \frac{\partial \mathbf{z}}{\partial \mathbf{y}} \right| = \mathcal{F}_1(\mathbf{y}), \\ m_1(\mathbf{y}) \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right| &= 1/V, \quad m_2(\mathbf{z}) \left| \frac{\partial \mathbf{z}}{\partial \mathbf{y}} \right| = m_1(\mathbf{y}), \end{aligned} \quad (5.53)$$

and $\mathbf{y} = h_1(\mathbf{x})$, $\mathbf{z} = h_2(\mathbf{y}) = h_2 \circ h_1(\mathbf{x})$. Thus $\tilde{\mathcal{I}}_\alpha^{(d)}(\mathcal{F})$ is invariant under the outcome-space reparametrization. In addition, if we restrict our consideration only to the class of transformations which have also differentiable inverse i.e., diffeomorphisms, we see from (5.52) and (5.53) that the information measure $\tilde{\mathcal{I}}_\alpha^{(d)}$ is invariant with respect to the group of diffeomorphisms. This fact was firstly realized by E.T. Jaynes in the context of Shannon's entropy [32]. As a matter of fact, when setting $\alpha = 1$ we obtain from (5.52) that

$$\begin{aligned} \tilde{\mathcal{H}}(\mathcal{F}) &= \lim_{\alpha \rightarrow 1} \frac{1}{(1-\alpha)} \log_2 \left(\int_{h(V)} d^d \mathbf{y} \left(\frac{\hat{\mathcal{F}}(\mathbf{y})}{m(\mathbf{y})} \right)^\alpha m(\mathbf{y}) \right) \\ &= - \int_{h(V)} d^d \mathbf{y} \hat{\mathcal{F}}(\mathbf{y}) \log_2 \left(\frac{\hat{\mathcal{F}}(\mathbf{y})}{m(\mathbf{y})} \right), \end{aligned} \quad (5.54)$$

which precisely coincides with Jaynes's finding [31,32]. Entropy (5.54) is also known as the Kullback-Leibler relative entropy.

APPENDIX F

In this appendix we derive relation (4.14). To start we must first identify \mathcal{E}_n . If we denote $N_l(a_i)$ as the number of boxes of size l needed to cover the multifractal with the singularity exponent a_i then $\mathcal{E}_n = \{\mathcal{E}_{nk}(a_i); k \in N_l(a_i), i \in \mathbb{N}\}$. Because of the scaling property we must set $\mathcal{E}_{nk}(a_i) = c_k(a_i)l^{a_i}$ with $c_k(a_i)$ weakly l dependent. In order to $\mathcal{I}_\alpha(\mathcal{E}_n)$ represent the "ground state" information we must require $c_k(a_i)$ to be a constant (i.e., $c_k(a_i, l) = c(l)$). This is so because in such a case our lack of information about the multifractal system (provided we comply with the scaling of probability) is clearly highest. This implies that $c = 1/\sum_i N_l(a_i)l^{a_i}$ as indeed

$$\sum_l \mathcal{E}_{nl} = \sum_i \sum_{k=1}^{N_l(a_i)} \mathcal{E}_{nk}(a_i) = \sum_i N_l(a_i) c l^{a_i} = 1. \quad (5.55)$$

Notice that c is weakly l dependent since $\sum_i N_l(a_i)l^{a_i} \sim l^{\tau(1)} = 1$. To proceed further we employ the multifractal measure (4.13). There $\mathcal{P}_n = \{p_{nk}\}$ is the discrete (integrated) probability distribution on the covering mesh. In case that the limit in (4.13) exists we may define the increment of $\mu_{\mathcal{P}}^{(\alpha)}(d; l)$ between a and $a + da$ in the small l limit as

$$d\mu_{\mathcal{P}}^{(\alpha)}(a) = \lim_{l \rightarrow 0} \sum_{l^{a+da} \lesssim p_{nk} \lesssim l^a} \frac{p_{nk}^\alpha}{l^\tau}. \quad (5.56)$$

Eq.(5.56) then implies that

$$\begin{aligned} \lim_{l \rightarrow 0} \log_2 \left(\sum_i \sum_k^{N_l(a_i)} p_{nk}^\alpha(a_i) \right) &\approx \log_2 \int_a d\mu_{\mathcal{P}}^{(\alpha)}(a) \\ &+ \tau(\alpha) \log_2 l, \end{aligned} \quad (5.57)$$

and so especially

$$\begin{aligned} \tilde{\mathcal{I}}_\alpha(\mu_{\mathcal{P}}) &\equiv \lim_{l \rightarrow 0} (\mathcal{I}_\alpha(\mathcal{P}_n) - \mathcal{I}_\alpha(\mathcal{E}_n)) \\ &= \frac{1}{(1-\alpha)} \log_2 \left(\frac{\int_a d\mu_{\mathcal{P}}^{(\alpha)}(a)}{\int_a d\mu_{\mathcal{E}}^{(\alpha)}(a)} \right). \end{aligned} \quad (5.58)$$

Under the condition that the integrals exist relation (5.58) represents a well defined (and finite) information measure. From the same reasons as in Section III we may conclude that $\tilde{\mathcal{I}}_\alpha(\mu_{\mathcal{P}})$ represents negentropy. Notice that similarly as before

$$\left. \int_a d\mu_{\mathcal{E}}^{(\alpha)}(a) \right|_{V=1} = 1. \quad (5.59)$$

This results from the fact that $\sum_i N_l(a_i)l^{\alpha a_i - \tau(\alpha)}$ is α independent in the small l limit. Actually,

$$\begin{aligned}
\frac{d}{d\alpha} K(\alpha) &\equiv \frac{d}{d\alpha} \sum_i N_l(a_i) l^{\alpha a_i - \tau(\alpha)} \\
&= \frac{d}{d\alpha} \int da \, n(a) l^{-f(a) + \alpha a - \tau(\alpha)} \\
&= \ln l \int da (a - a_0) n(a) l^{-f(a) + \alpha a - \tau(\alpha)} \\
&= 0 + \mathcal{O}\left(\frac{1}{(\ln l)^{3/2}}\right), \tag{5.60}
\end{aligned}$$

On the last line of (5.60) we have applied Laplace's formula of the asymptotic calculus [38]. Eq.(5.60) confirms our previous assertion as it assures that the vanishing of $dK(\alpha)/d\alpha$ at $l \rightarrow 0$ is at least as large as that of $1/(\ln l)^{3/2}$. The consequence of this is that

$$\begin{aligned}
\lim_{l \rightarrow 0} \frac{\sum_i N_l(a_i) l^{\alpha a_i - \tau(\alpha)}}{(\sum_i N_l(a_i) l^{a_i})^\alpha} &= \frac{K(\alpha)}{(K(1))^\alpha} = (K(1))^{1-\alpha} \\
&= \frac{K(0)}{(K(1))^\alpha} = \frac{V}{(K(1))^\alpha} = \frac{1}{(K(1))^\alpha}. \tag{5.61}
\end{aligned}$$

The latter implies that $K(1) = 1$ and ergo (5.59) holds.

APPENDIX G

We show here an alternative way to obtain the real inverse formula for Eq.(4.19). Let us start with the following observation:

$$\mathcal{F}_{\mathcal{P}}(x) = \sum_{-\log_2 p_k < x} p_k = \sum_l p_l \theta(\log_2 p_l + x). \tag{5.62}$$

Using the limit representation of the step function $\theta(x)$;

$$\theta(x) = \lim_{\varepsilon \rightarrow 0_+} \exp\left(-2^{-\frac{x}{\varepsilon}}\right),$$

together with the functional relation

$$\begin{aligned}
\theta(\log_2 p_l + x) &= \theta(x) \theta\left(\frac{\log_2 p_l}{x} + 1\right) \\
&+ \theta(-x) \theta\left(-\frac{\log_2 p_l}{x} - 1\right) \\
&= \theta(x) - \varepsilon(x) \theta\left(-\frac{\log_2 p_l}{x} - 1\right), \tag{5.63}
\end{aligned}$$

we may rewrite (5.62) as

$$\mathcal{F}_{\mathcal{P}}(x) = \theta(x) - \lim_{\varepsilon \rightarrow 0_+} \varepsilon(x) \sum_{n=0}^{\infty} \frac{(-1)^n 2^{\frac{n}{\varepsilon}}}{n!} 2^{-\frac{n}{\varepsilon} \mathcal{I}_{(n/\varepsilon x + 1)}}. \tag{5.64}$$

or equivalently

$$\mathcal{F}_{\mathcal{P}}^c(x) \approx \varepsilon(x) \sum_{n=0}^{\infty} \frac{(-1)^n 2^{\Lambda n}}{n!} 2^{-\frac{n\Lambda}{x} \mathcal{I}_{(\Lambda n/x + 1)}}. \tag{5.65}$$

Here the complementary information–distribution function of \mathcal{P}

$$\mathcal{F}_{\mathcal{P}}^c(x) \equiv \theta(x) - \mathcal{F}_{\mathcal{P}}(x) = \sum_{-\log_2 p_k \geq x \geq 0} p_k,$$

was defined. The regulator $\Lambda \sim 1/\varepsilon$. Note that because $x \in [0, +\infty)$ we have that $\alpha \in [1, +\infty)$. This is in the agreement with the analysis based on the Widder–Stiltjes inverse formula.

APPENDIX H

In this appendix we derive the reconstruction theorem for THC entropy. Starting with Eq.(4.20) we may write

$$\begin{aligned}
\mathcal{F}_{\mathcal{P}}(x) &= \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp \frac{e^{px} e^{-p\mathcal{I}_{\alpha}(\mathcal{P})}}{p} \\
&= -\frac{1}{\ln(4)\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{px} \mathcal{S}_{\alpha}(\mathcal{P}) + \theta(x) \tag{5.66}
\end{aligned}$$

where the step function $\theta(x)$ was added and subtracted and the Bromwich representation

$$\theta(x) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp \frac{e^{px}}{p},$$

was used. As a result we obtain

$$\mathcal{F}_{\mathcal{P}}^c(x) = \frac{1}{\ln(4)\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{px} \mathcal{S}_{\alpha}(\mathcal{P}). \tag{5.67}$$

The inverse Laplace–Stiltjes transformation then gives

$$\mathcal{S}_{\alpha}(\mathcal{P}) = \frac{1}{(\alpha-1)} \int_{x=0}^{\infty} 2^{(1-\alpha)x} d\mathcal{F}_{\mathcal{P}}^c(x). \tag{5.68}$$

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